DEMICONTINUITY, HEMICONTINUITY AND MONOTONICITY

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Recently the notions of monotone, demicontinuous and hemicontinuous functions have been introduced in connection with nonlinear problems in functional analysis (Browder [1; 2; 3; 4; 5], Minty [6; 7; 8]). The object of the present note is to show that under rather general conditions, hemicontinuity is equivalent to demicontinuity for monotone functions.

Let $X$ be a (real or complex) Banach space and $X^*$ its adjoint space as the set of all bounded conjugate-linear functionals on $X$. The value of $f \in X^*$ at $u \in X$ is denoted by $(f, u)$. We use the notations $\to$ and $\rightharpoonup$ for strong convergence in $X$ (or in $X^*$ or in the set of real numbers) and weak* convergence in $X^*$, respectively.

Let $G$ be a function from $X$ to $X^*$ with domain $D=D(G) \subset X$. $G$ is said to be demicontinuous if $u_n \in D$, $n=1, 2, 3, \ldots$, $u \in D$ and $u_n \rightharpoonup u$ imply $G(u_n) \rightharpoonup Gu$. $G$ is hemicontinuous if $u \in D$, $v \in X$ and $u + t_n v \in D$, where $t_n$ is a sequence of positive numbers such that $t_n \to 0$, imply $G(u + t_n v) \to Gu$. We shall say that $G$ is locally bounded if $u_n \in D$, $u \in D$ and $u_n \to u$ imply that $Gu_n$ is bounded. Obviously a demicontinuous function is hemicontinuous and locally bounded.

$G$ is said to be monotone if $\Re(Gu - Gv, u - v) \geq 0$ for $u, v \in D$.

These definitions may be void if $D$ is too arbitrary. In what follows we shall assume that $D$ is quasi-dense. By this we mean that for each $u \in D$ there is a dense subset $M_u$ of $X$ such that for each $v \in M_u$, $u + tv \in D$ for sufficiently small $t>0$ (the smallness of $t$ depending on $v$). Thus any open subset of $X$ as well as any dense linear manifold of $X$ is quasi-dense.

**Theorem 1.** Let $G$ be a monotone function from $X$ to $X^*$ with a quasi-dense domain $D$. Then $G$ is demicontinuous if and only if it is hemicontinuous and locally bounded.

**Proof.** By the remark given above, it suffices to prove the "if" part. Suppose $G$ is hemicontinuous and locally bounded. Let $u_n \to u$, $u_n, u \in D$. We have to show that $Gu_n \rightharpoonup Gu$. Obviously we may assume that $u_n \neq u$.

Let $M_u$ be the dense subset of $X$ used in the definition of $D$ being quasi-dense. Let $v \in M_u$ and $t_n = \|u_n - v\|^{1/2}$. Then $t_n > 0$, $t_n \to 0$, $w_n = u + t_nv \in D$ for sufficiently large $n$ and
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(1) \( Gw_n \rightarrow Gu \).

Now the monotonicity of \( G \) implies

(2) \( 0 \leq \text{Re}(Gu_n - Gw_n, u_n - w_n) = \text{Re}(Gu_n - Gw_n, u_n - u - t_n v) \).

\( Gu_n \) is bounded since \( G \) is locally bounded. \( Gw_n \) is bounded by (1). Hence

\[
\lim_{t_n \to 0} \frac{1}{t_n} \text{Re}(Gu_n - Gw_n, u_n - u) \to 0
\]

because \( \|t_n^{-1}(u_n - u)\| = t_n \to 0 \). Also \((Gw_n, v) \to (Gu, v)\) by (1). Dividing (2) by \( t_n \) and letting \( n \to \infty \), we thus obtain

(3) \( \lim \inf \text{Re}(Gu_n - Gu, -v) \geq 0 \).

(3) is true for any \( v \in M_u \). Since \( M_u \) is dense in \( X \) and \( Gu_n \) is bounded in \( X^* \), it follows that (3) is true for every \( v \in X \). Replacing \( v \) by \( -v \) (and also by \( \pm iv \) if \( X \) is complex) and putting the results together, we obtain

\[
\lim (Gu_n - Gu, v) = 0, \quad v \in X.
\]

This proves that \( Gu_n \to Gu \), q.e.d.

REMARK 1. Theorem 1 shows that a monotone hemicontinuous function that maps bounded sets into bounded sets is a notion stronger than a monotone demicontinuous function. (Such functions are considered in \([2-III]\) and \([5]\).)

REMARK 2. It is not clear whether the assumption of local boundedness in Theorem 1 can be eliminated. But this is the case if \( X \) is finite-dimensional. We have namely

**Theorem 2.** Let \( X \) be a finite-dimensional Banach space. Let \( G \) be a monotone function from \( X \) to \( X^* \) with a quasi-dense domain \( D \). Then \( G \) is continuous if and only if it is hemicontinuous.

**Proof.** Since continuity and demicontinuity are equivalent when \( X \) is finite-dimensional, it suffices to show that \( G \) is locally bounded if it is hemicontinuous; then the result follows from Theorem 1.

Suppose that \( G \) is hemicontinuous but not locally bounded. Then there is a \( u \in D \) and a sequence \( u_n \in D \) such that \( u_n \to u \) but \( Gu_n \) is unbounded. We may assume without loss of generality that \( \|Gu_n\| = s_n \to \infty \). Let \( M_u \) be as above and let \( v \in M_u \). Take a \( t > 0 \) so small that \( u + tv \in D \). Then by monotonicity

\[
0 \leq s_n^{-1} \text{Re}(Gu_n - G(u + tv), u_n - u - tv)
\]

(4) \( = \text{Re}(s_n^{-1}Gu_n - s_n^{-1}G(u + tv), u_n - u - tv) \).
Now $s_n^{-1}G(u + tv) \to 0$, $u_n - u \to 0$ and $s_n^{-1}Gu_n$ is bounded. On dividing (4) by $t > 0$ and letting $n \to \infty$, we thus obtain

$$\lim \inf \Re(s_n^{-1}Gu_n, -v) \geq 0.$$  

As in the proof of Theorem 1, this leads to the result that $s_n^{-1}Gu_n \to 0$. But this is a contradiction, for $\|s_n^{-1}Gu_n\| = 1$ and weak* convergence is equivalent to strong convergence.

**BIBLIOGRAPHY**


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