THE CLOSING LEMMA AND STRUCTURAL STABILITY

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Introduction. Consider a differentiable n-manifold $M$. Let $\mathfrak{X} = \mathfrak{X}(M)$ be the space of all $C^1$ tangent vector fields on $M$ under a $C^1$ topology [1]. Each $X \in \mathfrak{X}$ induces a $C^1$-flow on $M$ called the $X$-flow. Let $d$ be a metric on $M$ and let $\epsilon$ be positive. Two flows are homeomorphic if there is a homeomorphism $h$ of $M$ onto itself taking the trajectories of one flow onto those of the other; the two flows are $\epsilon$-homeomorphic if $h$ can be chosen so that $d(h(p), p) < \epsilon$ for all $p \in M$. $X$ is said to be structurally stable if, given $\epsilon > 0$, there then exists a neighborhood $\mathfrak{U}$ of $X$ in $\mathfrak{X}$ such that for each $Y \in \mathfrak{U}$ the $Y$-flow is $\epsilon$-homeomorphic to the $X$-flow. Let us say that $X$ is crudely structurally stable if we drop the $\epsilon$ condition: $X$ is crudely structurally stable if there exists a neighborhood $\mathfrak{U}$ of $X$ in $\mathfrak{X}$ such that $Y \in \mathfrak{U}$ implies that the $Y$-flow is homeomorphic to the $X$-flow. Let $\Sigma$ denote those $X$ in $\mathfrak{X}$ which are structurally stable and let $\Sigma_\epsilon$ denote those $X$ in $\mathfrak{X}$ which are crudely structurally stable, obviously $\Sigma \subseteq \Sigma_\epsilon$. The problem of structural stability theory is to characterize $\Sigma$ and $\Sigma_\epsilon$ and to study the topological relation of $\Sigma$ and $\Sigma_\epsilon$ to $\mathfrak{X}$.

The most comprehensive results in structural stability theory are due to M. Peixoto [2], [3], [4] who has shown, when $M$ is a compact 2-manifold, that $\Sigma = \Sigma_\epsilon$, $\Sigma = \mathfrak{X}$, and that the fields in $\Sigma$ are characterized completely as the fields with "generic" induced flows.

Related to the problem of structural stability is the following conjecture:

**Closing Lemma.** If the $X$-flow has a nontrivial recurrent trajectory through some $p \in M$ and if $\mathfrak{U}$ is any neighborhood of $X$ in $\mathfrak{X}$ then there exists $Y \in \mathfrak{U}$ such that the $Y$-flow has a closed orbit through $p$.

(Recall that a trajectory is nontrivially recurrent if it is contained in its $\alpha$- or in its $\omega$-limit set without being a closed orbit or a stationary point.)

**Results concerning the Closing Lemma.** M. Peixoto [4] has proved the Closing Lemma in the case that $M$ is the 2-torus and $X$ has no
singularities. We prove the following two forms of the Closing Lemma. (Our proofs, however, are invalid for a $C^r$ topology on $\mathcal{X}$, $r > 1$.)

**Theorem 1.** Let $M$ be any differentiable 2-manifold and let $X \in \mathcal{X}$ have a nontrivial recurrent trajectory through $p \in M$. Let $U$ be an arbitrarily small coordinate neighborhood of $p$ in $M$ and let $\epsilon > 0$ be given. Then there exists $\Delta \in \mathcal{X}$ such that

(a) $\Delta$ vanishes on $M - U$.

(b) The $C^1$ size of $\Delta$ respecting the coordinates of $U$ is less than $\epsilon$.

(c) $Y = X + \Delta$ has a closed orbit through $p$.

**Theorem 2.** Let $M$ be a compact $n$-manifold and let a Riemannian metric be put on $M$ so that the norm of each linear transformation $L: T_x(M) \to T_x(M)$ is defined. Suppose that $X \in \mathcal{X}$ induces a flow $\phi$ which has a nontrivial recurrent trajectory through $p \in M$. Define $J(t, x): T_x(M) \to T_{\phi(t, x)}(M)$ to be the jacobian isomorphism of tangent spaces induced by $x \to \phi(t, x)$. Suppose that $\epsilon > 0$ is given and that

$$\lim_{t \to \infty} \frac{1}{t} \| J^{-1}(t, p) \| = 0.$$  

Then there exists $\Delta \in \mathcal{X}$ such that the $C^1$ size of $\Delta$ is less than $\epsilon$ and $Y = X + \Delta$ has a closed orbit through $p$.

Where $M$ is compact, all Riemannian metrics are equivalent and so Theorem 2 does not depend on the choice of Riemannian metric.

**Definition.** Let $X$ be in $\mathcal{X}(M)$ for a differentiable $n$-manifold $M$. A flow-box for $X$ at $p \in M$ is a coordinate neighborhood $U$ of $p$ in $M$ such that in terms of the coordinates $(u_1, \ldots, u_n)$ of $U$, $u_i(p) = 0$ for $i = 1, 2, \ldots, n$ and

$$X_u = \left( \frac{\partial}{\partial u^i} \right)_u$$  

for all $u$ in $U$.

If $X_p \neq 0$, then it is well known that a flow-box for $X$ at $p$ exists.

The following lemma is the principal tool used to prove Theorems 1 and 2.

**Lemma.** Let $\epsilon > 0$ and $0 < \delta < 1$ be given. Let $M$ be a differentiable $n$-manifold and let $X \in \mathcal{X}$ induce the flow $\phi$. Suppose that $X$ does not vanish at $p^* \in M$ and let $U$ be a flow-box for $X$ at $p^*$. Let

$$\Pi = \{(0, u_2, u_3, \ldots, u_n) \in U\}.$$  

Suppose that $P$ is a subset of $\Pi$ such that arbitrarily near $p^*$ there are distinct points of $P$ lying on a common $\phi$-trajectory (e.g., let $P = \mu \cap \Pi$
and let \( p^* \in \mu \cap \Pi \) where \( \mu \) is a nontrivial recurrent \( \phi \)-trajectory. Then there exist points \( p \) and \( q \) of \( P \) such that

\[
|p - p^*| < \varepsilon, \\
|q - p^*| < \varepsilon,
\]

(a)

\[
\phi(t^*, p) = q \text{ for some } t^* > 0,
\]

and

(b) \( \varepsilon \) If \( r = \phi(t', p) \in P \) for some \( t' < t^* \),

\[
|p - r| > \delta |p - q| \text{ and } |q - r| > \delta |p - q|,
\]

where \( |x - y| \) denotes the distance between \( x \) and \( y \) respecting the coordinates of \( U \).

The proof of this lemma is easy. Just take a \( p_0 \) and \( q_0 \) in \( P \) obeying (a) where \( \varepsilon \) has been replaced by the smaller constant \( \frac{1}{2}(1 - \delta) \cdot \varepsilon \) and where \( t^* \) is called \( t_0 \). If (b) fails to be true for some \( r = \phi(t', p_0) \), then suppose that \( |q_0 - r| \leq \delta |p_0 - q_0| \). Replace \( q_0 \) by \( r \) and regard the pair \((p_0, r)\) instead of the pair \((p_0, q_0)\). Call \((p_0, r) = (p_1, q_1)\). Proceed similarly if \( |q_0 - r| > \delta |p_0 - q_0| \) but \( |p_0 - r| \leq \delta |p_0 - q_0| \) to get \((p_1, q_1) = (r, q_0)\). Proceed with \((p_1, q_1)\) as was done with \((p_0, q_0)\), getting, thereby, a sequence \((p_k, q_k)\) \( k = 1, 2, \ldots \). The process ends at a finite step \((p_m, q_m)\) because \( \phi(t, p) \) crosses \( \Pi \) at most a finite number of times for \( 0 \leq t \leq t_0 \). The pair \((p_m, q_m)\) satisfies (b) by construction. It also satisfies (a) because

\[
|p^* - p_m| \leq \sum_{i=1}^{m} \max( |p_i - p_{i-1}|, |q_i - q_{i-1}| ) + |p_0 - p^*|
\]

\[
\leq \sum_{i=1}^{m} \delta^i |p_0 - q_0| + |p_0 - p^*|
\]

\[
< |p_0 - q_0| \cdot \frac{1}{1 - \delta} + |p_0 - p^*|
\]

\[
< \frac{\varepsilon \cdot (1 - \delta)}{2 \cdot (1 - \delta)} < \varepsilon.
\]

Similarly \( |p^* - q_m| < \varepsilon \).

As a consequence of Theorem 1, M. Peixoto's paper [4] can be shortened considerably. The methods used to prove Theorem 1 can also be used to solve the following problem.

Suppose that \( M = S^2 \), \( X \subset \mathcal{X}(S^2) \), and that the \( X \)-flow has a closed orbit \( \gamma \) which is isolated but unstable. Suppose there are \( n \) generic saddle points \( p_1, p_2, \ldots, p_n \) outside \( \gamma \) and \( n \) more generic saddle points \( q_1, q_2, \ldots, q_n \) inside \( \gamma \) such that one separatrix from each \( p_i \)
has $\gamma$ as an $\omega$-limit and one separatrix from each $q_i$ has $\gamma$ as an $\alpha$-limit point. The problem is to find an arbitrarily $C^1$ small $\Delta \in \mathcal{X}$ such that for $Y = X + \Delta$, the $Y$-flow "joins the $p_i$'s to the $q_j$'s." That is, each $p_i$ should have a $Y$-separatrix $\sigma_i$ which is also a $Y$-separatrix of some $q_j$. When $\Delta$ is sufficiently $C^1$ small, it is easily seen that the same $q_j$ cannot be joined to two different $p_i$'s. M. Peixoto [4] has solved this problem for $n = 1$. The problem for $n \geq 2$ is related to an investigation of "higher order structural stability" at present being completed by G. Sottomayor. Sottomayor wishes $\Delta$ to be $C^\infty$ small, but—as in the Closing Lemma itself—our methods only produce perturbations which are $C^1$ small.

I hope that Theorem 2 will yield as a corollary that distal minimal nontrivial recurrent flows on compact differentiable manifolds may be closed by arbitrarily $C^1$ small perturbations $\Delta$. It would suffice to prove that for some $p \in M$, \[\|J^{-1}(t, p)\|\] is bounded as $t \to \infty$ where $J$ is the Jacobian isomorphism induced as in Theorem 2. Roughly, the reason this should be true is that $\|J^{-1}\|$ is a measure of how fast the flow contracts and distal flows don't contract too much.

Finally, we inspect two examples related to the theory of structural stability for noncompact 2-manifolds. First we show that for $M = \mathbb{R}^2$, $\Sigma_e \neq \Sigma$. Second, following M. L. Peixoto, we see that there exists a nonvanishing $X \in \mathcal{X} (\mathbb{R}^2)$ which is not in $\Sigma_e$. This shows that it will probably be quite difficult to characterize the elements of $\Sigma$ and $\Sigma_e$ for noncompact 2-manifolds.

In a sense, this is unfortunate because Theorem 1 holds for noncompact differentiable 2-manifolds and one might hope to use it to try to generalize M. Peixoto's characterization theorem [4] to the noncompact case. In particular one would hope to show that $X \in \Sigma_e$ if the $X$-flow has a nontrivial recurrent trajectory. I can prove this if $M$ has finite genus but if $M$ has infinite genus, I can prove it only by using the following

**Conjecture.** Suppose that $M$ is a differentiable 2-manifold and that $X \in \Sigma_e (M)$. Let $\Gamma$ be the union of all the closed orbits of the $X$-flow. Then $\Gamma$ is closed in $M$.

**References**