A TWO-DIMENSIONAL SINGULAR INTEGRAL EQUATION OF DIFFRACTION THEORY

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Communicated by A. Zygmund, February 28, 1964

The formulation of a problem in diffraction theory has led us to consider the two-dimensional singular integral equation

\[ \iint_{Q_{13}} f(t_1, t_2) k\left( |x_1 - t_1|, |x_2 - t_2| \right) dt_1 dt_2 = 0 \]

where: \( Q_{13} \) denotes the union of quadrants I, III; \( f \) is unknown, but must vanish on quadrants II, IV; the equation is valid only for \( x = (x_1, x_2) \) in \( Q_{13} \); and \( k \) is the diffraction-theoretic kernel

\[ k(x) = - (4\pi r)^{-1} \exp(-i\beta r) \]

with \( r = (x_1^2 + x_2^2)^{1/2} \) and \( \beta \) complex \([\text{Im}(\beta) < 0]\).

In earlier physical investigations, we had encountered variants of (1) in which the domains of integration and validity were (a) two contiguous quadrants (see [4]) or (b) one quadrant (see [5], [7]); and it is clear that the equation over three quadrants may be treated by methods applicable to the complementary case (b). Thus, the present study of (1) completes a theory of two-dimensional convolution-type equations with the diffraction-theoretic kernel \( k \) over quadrants of the \( x_1x_2 \)-plane. Since these equations generalize the one-dimensional convolution-type on the half-line (i.e., the classical equation of Wiener and Hopf [9]), the theory is a partial extension of Wiener and Hopf's ideas from one to two dimensions.

Our analysis may be divided into three parts:

I. Preparatory. The integral equation (1) is extended to \( X \), the whole \( x_1x_2 \)-plane, whereupon the left side becomes a convolution (Wiener's "Faltung") while the right side \( h(x) \) is defined (but not known) on \( X - Q_{13} \), and \( h = 0 \) on \( Q_{13} \):

\[ f \ast k = h. \]

The two-dimensional Laplace transformation ([1], Chapter VI of [2], or [3]) then maps (3) into the transform equation (capital letters denote transforms; \( w = (w_1, w_2) \) denotes a point in a product-space

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1 This work was supported in part by the Office of Naval Research, under Contract No. Nonr 3360 (01).
of two complex variables, with \( w_j = u_j + iv_j \) and \( j = 1, 2 \) here and in what follows):

\[
F(w)K(w) = H(w)
\]  

which is to be solved for the two unknown functions \( F, H \).

It is known that \( K(w) = (i/2)(w_1 + w_2 + \beta^2)^{-1/2} \); thus, \( K \) is analytic for \( u = (u_1, u_2) \) in a product domain \( B: W_1 \times W_2 \), where the \( W_j \) are vertical strips interior to \( |u_j| \leq |\text{Im} \beta| \). Assume next that \( F \) is a distribution (cf. \([3, \text{Proposition 4.2, p. 14}]\)) representable as

\[
F(w) = P(w)G(w),
\]

where \( P \) is a polynomial, and

\[
G(w) = (\varphi_1 + \varphi_2)g(x),
\]

the restricted Laplace transform \( \varphi_n \) being defined by an integral over the closed \( n \)th quadrant:

\[
\varphi_n g = \iint_{Q_n} g(x) \exp(-w \cdot x)dx_1dx_2
\]

\( (w \cdot x = w_1x_1 + w_2x_2) \). Finally, let \( g(x) \exp(-w \cdot x) \) be of bounded \( L_2 \) norm over \( Q_1, Q_3 \) for \( u \) in the respective domains

\[
\begin{align*}
(8.1) & \quad C_1: u_j > b_j > 0, \\
(8.2) & \quad C_3: u_j < -b_j < 0
\end{align*}
\]

with \( C_1 \cap C_3 \) empty, as indicated in (8.1), (8.2), while \( C_1 \cap B \) and \( C_3 \cap B \) are nonempty.

[Remark. If \( C_1 \cap C_3 \) is nonempty, \( G \equiv 0 \). The same situation is noted in \([8]\), whose subject is the presently relevant one of characterizing two-variable Laplace transforms of functions with support in \( Q_{13} \). Some of the considerations which arise are exemplified in the proof of Lemma II.2 below.] It may then be shown that:

\begin{description}
\item[Statement 1.1.] \( F(w) \) and \( H(w) \) as well as \( K(w) \) are analytic for \( u \) in \( B \).
\item[Statement 1.2.] \( F(w) = F_1(w) + F_3(w), \ H(w) = H_3(w) + H_4(w), \)
\end{description}

where: subscripts \( n \) \( (n=1, 2, 3, 4) \) denote functions analytic for \( u \) in \( (B, n) \), while \( (B, n) \) signifies the convex closure of \( B \) and the \( n \)th quadrant of the \( u \)-plane.

II. Factorization is the key step as in \([9]\), but the single factorization lemma of Wiener and Hopf's one-dimensional theory must now be replaced by two lemmas:

\begin{description}
\item[Lemma II.1.] \( K(w) \) may be uniquely expressed as the product of four
functions $M_n(w)$, analytic and nonzero for $u$ in $(B, n)$. [This is shown, and the $M_n(w)$ are explicitly calculated, in [5, §5].]

**Lemma 11.2.** $K_{13}(w) = M_1(w)M_3(w)$ and $K_{24}(w) = M_3(w)M_4(w)$ are analytic and nonzero in the respective pairs of disjoint $u$-domains ($-\infty < v_j < +\infty$ throughout) $q_1$, $q_3$ and $q_2$, $q_4$, where we have (for any $\delta > 0$)

\[(9)\] $q_1: (u_1 + u_2) \geq (|\text{Im}(\beta)| + \delta) \quad (u_j > 0)$

and $q_2, q_3, q_4$ are successive reflections of $q_1$ in $u_2 > 0, u_1 < 0, u_2 < 0$.

**Proof.** Introduce the function

\[\phi(x) = N_0(\beta r), \quad x \in Q_{13}\]
\[= 0, \quad x \in Q_{24}\]

where $N_0(\beta r)$ is the Bessel function of the second kind (Neumann's function). The image of $\phi$ under two-dimensional Laplace transformation is, as shown in [6],

\[\phi(w) = 2i(w_1^2 + w_2^2 + \beta^2)^{-1}[i + w_1s_2s_3 + w_2s_1s_3],\]

with

\[(11.1)\] $s_j = (w_j^2 + \beta^2)^{-1/2}$ \quad ($s_j = \beta^{-1}$ at $w_j = 0$)

\[(11.2)\] $S_j = (2i\pi^{-1}) \log [\beta^{-1}(w_j + s_j^{-1})]\]

and, significantly,

\[\mathcal{L}\phi(x) = 2\mathcal{L}_1\phi(x) = 2\mathcal{L}_3\phi(x).\]

As appears from (12), $\Phi(w) = \mathcal{L}\phi(x)$ is analytic for $u$ in $q_1$ and for $u$ in $q_2$. The same is true of $K_{13}(w)$, since it may be shown (by the reasoning of [5, §5]) that

\[K_{13}(w) = \exp\left[\int_0^\beta \Phi(w; \beta) d\beta\right]\]

where $\Phi$ is written $\Phi(w; \beta)$ to emphasize the dependence on $\beta$, and it is understood that $\beta$ is allowed to vary in a small neighborhood of its fixed value for purposes of the integration. The assertion for $K_{13}$ is therefore proved, and the proof for $K_{24}$ follows by symmetry.

**III. Solutions of the transform equation and the integral equation.**

Two theorems may now be proved without difficulty (the first requires only verification):
Theorem III.1. The transform equation (4) has the solutions

\begin{align}
F(w) &= c_0[K_{13}(w)]^{-1}, \\
H(w) &= c_0K_{24}(w)
\end{align}

where $c_0$ is an arbitrary constant [the same arbitrary constant in (14.1), (14.2)].

Theorem III.2. The functions $F, H$ of (14.1), (14.2) possess two-variable Laplace inverses $f(x), h(x)$, and the latter pair are literal solutions of (3). [The function $f(x)$ is of course a literal solution of (1), as well as of the extended equation (3).]

References


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