THE RECURSIVE EQUIVALENCE TYPE OF
A CLASS OF SETS

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1. Introduction. Let us consider non-negative integers (numbers),
collections of numbers (sets) and collections of sets (classes). The
letters $\epsilon$ and $\varnothing$ stand for the set of all numbers and the empty set
of numbers respectively. We write $\subseteq$ for inclusion, proper or improper.
A mapping from a subset of $\epsilon$ into $\epsilon$ is called a function; if $f$ is a func-
tion, we denote its domain and its range by $\delta f$ and $\rho f$ respectively. Let
a class of mutually disjoint nonempty sets be called an md-class; such
a class is therefore countable, i.e., finite or denumerable. We recall
that the recursive equivalence type (abbreviated: RET) of a set $\alpha$, denoted by $\text{Req}(\alpha)$, is defined [1, p. 69] as the class of all sets which
are recursively equivalent to $\alpha$. We wish to consider the problem:
"How can we define the RET of an md-class in a natural manner?"
Throughout this note $\mathcal{S}$ stands for an md-class and $\sigma$ for the union of
all sets in $\mathcal{S}$; for every $x \in \sigma$ we denote the unique set $\alpha$ such that
$x \in \alpha \subseteq \mathcal{S}$ by $\alpha_x$.

DEFINITIONS. A set $\gamma$ is a choice set of $\mathcal{S}$, if

1. $\gamma \subseteq \sigma$,
2. $\gamma$ has exactly one element in common with each set in $\mathcal{S}$.

The set $\gamma$ is a good choice set of $\mathcal{S}$ (abbreviated: gc-set), if it also
satisfies

3. there exists a partial recursive function $p(x)$ such that $\sigma \subseteq \delta p$
and $(\forall x)[x \in \sigma \Rightarrow p(x) \in \gamma \cdot \alpha_x]$.

Consider the special case that the md-class $\mathcal{S}$ is a finite class of
finite sets. Then

(a) every choice set of $\mathcal{S}$ is a good choice set,
(b) every two choice sets of $\mathcal{S}$ are recursively equivalent,
(c) every two good choice sets of $\mathcal{S}$ are recursively equivalent.

If the md-class $\mathcal{S}$ is infinite, (a) and (b) need no longer be true.
For let $\mathcal{S}$ contain infinitely many sets of cardinality $\geq 2$, e.g.,
$\mathcal{S} = \{(0, 1), (2, 3), (4, 5), \ldots \}$. Then $\mathcal{S}$ has $\epsilon$ choice sets. Every good
choice set of $\mathcal{S}$ has the form $p(\sigma)$ for some partial recursive function
$p(x)$, hence $\mathcal{S}$ has at most $\aleph_0$ good choice sets and (a) is false. Every
nonzero RET contains exactly $\aleph_0$ sets; the $\epsilon$ choice sets of $\mathcal{S}$ can
therefore not all be recursively equivalent and (b) is false. On the

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other hand, (c) still holds. For we have

**Proposition P1.** Every two good choice sets of an md-class are recursively equivalent.

Note that (a) does not even hold for every finite class consisting of two infinite sets. For let \( S = (\tau, \tau') \), where \( \tau \) and \( \tau' \) are complementary immune sets. Then \( S \) has denumerably many choice sets, but if \( S \) had a good choice set, \( \tau \) and \( \tau' \) would be recursive. For every md-class \( S \) we write \( \xi(S) \) for the class of all gc-sets of \( S \). If \( \xi(S) \) is nonempty, \( S \) is called a gc-class. The class \( (\tau, \tau') \) mentioned above is an example of an md-class which is not a gc-class. P1 enables us to give the

**Definition.** For any gc-class \( S \),

\[
\text{RET}(S) = \text{Req}(\gamma), \quad \text{for any } \gamma \in \xi(S).
\]

If \( S \) is a finite md-class of finite sets, \( S \) is a gc-class and \( \text{RET}(S) \) equals the cardinality of \( S \). We need not exclude the trivial case that \( S \) is empty, for then \( \xi(S) \) contains exactly one set, namely \( 0 \).

2. **Elementary properties.** The sets \( \alpha_0, \cdots, \alpha_n \) are separable if there exist mutually disjoint r.e. sets \( \beta_0, \cdots, \beta_n \) such that \( \alpha_i \subseteq \beta_i \) for \( 0 \leq i \leq n \). We write \( \alpha_0 \mathrel{|} \alpha_1 \) if \( \alpha_0 \) and \( \alpha_1 \) are separable.

**Proposition P2.** The finite md-class \( S = (\alpha_0, \cdots, \alpha_n) \) is a gc-class if and only if \( \alpha_0, \cdots, \alpha_n \) are separable; if \( S \) is a gc-class, each choice set of \( S \) is a gc-set and \( \text{RET}(S) \) equals the cardinality of \( S \).

A gc-class is called isolated if each (or equivalently, at least one) of its gc-sets is isolated. In other words, a gc-class is isolated if its RET is an isol. For every nonempty gc-class \( S \) we have: \( \sigma \) is a finite set if and only if \( S \) is a finite class of finite sets. Similarly,

**Proposition P3.** Let \( S \) be a nonempty gc-class. Then \( \sigma \) is an isolated set if and only if \( S \) is an isolated class of isolated sets.

Two classes \( S_1 \) and \( S_2 \) with unions \( \sigma_1 \) and \( \sigma_2 \) respectively are separable if \( \sigma_1 \mathrel{|} \sigma_2 \). For any two classes \( A \) and \( B \) we write

\[
A \times B = \{ j(\alpha \times \beta) \mid \alpha \in A \text{ and } \beta \in B \},
\]

where \( j(x, y) = x + (x+y)(x+y+1)/2 \).

**Proposition P4.** Let \( S_1 \) and \( S_2 \) be separable md-classes. Then \( S_1 \cup S_2 \) is an md-class and

(a) \( S_1 \cup S_2 \) is a gc-class if and only if both \( S_1 \) and \( S_2 \) are gc-classes,

(b) if \( S_1 \cup S_2 \) is a gc-class, \( \text{RET}(S_1 \cup S_2) = \text{RET}(S_1) + \text{RET}(S_2) \).
PROPOSITION P5. Let $S_1$ and $S_2$ be nonempty md-classes. Then $S_1 \times S_2$ is a nonempty md-class and

(a) $S_1 \times S_2$ is a gc-class if and only if both $S_1$ and $S_2$ are gc-classes,
(b) if $S_1 \times S_2$ is a gc-class, $\text{RET}(S_1 \times S_2) = \text{RET}(S_1) \cdot \text{RET}(S_2)$.

3. The class $\text{Bin}(\alpha)$. Let $\{\rho_n\}$ be the canonical enumeration of the class of all finite sets [2, p. 81] and $r_n =$ cardinality of $\rho_n$. For any set $\alpha$ and any number $k$ we write

$$C(\alpha, k) = \{n \mid \rho_n \subset \alpha \text{ and } r_n = k\}, \quad \text{Bin}(\alpha) = \{C(\alpha, k) \mid k \geq 1\}.$$ 

Note that $\text{Bin}(\alpha)$ is an md-class for any set $\alpha$; if $\alpha$ is a finite set of cardinality $n$, the members of $\text{Bin}(\alpha)$ are separable and $\text{Bin}(\alpha)$ is a gc-class with $n$ as cardinality and RET. For any infinite set $\alpha$, $\text{Bin}(\alpha)$ is a denumerable md-class of infinite sets; the next proposition tells us when $\text{Bin}(\alpha)$ is a gc-class. We write $\text{Req}(\epsilon) = R$ and refer to [2, pp. 80, 84] for the definition of a regressive set and a regressive isol.

PROPOSITION P6. Let $\alpha$ be infinite and $A = \text{Req}(\alpha)$. Then

(a) if $\alpha$ has an infinite r.e. subset, $\text{Bin}(\alpha)$ is a gc-class of RET $R$,
(b) if $\alpha$ is a regressive set, $\text{Bin}(\alpha)$ is a gc-class of RET $A$,
(c) if $\alpha$ is immune, but not regressive, $\text{Bin}(\alpha)$ is not a gc-class.

It follows that among the $c$ existing md-classes of immune sets, exactly $c$ are gc-classes and exactly $c$ are not. It is shown in [3] that though the collection $\Delta_R$ of all regressive isols is not closed under addition one multiplication, one can extend the $\text{min}(\langle x, y \rangle)$ function from $\mathcal{E}$ into $\epsilon$ in a natural manner to a $\text{min}(X, Y)$ function from $\Delta_R$ into $\Delta_R$. However, $\text{min}(X, Y)$ need no longer assume one of the values $X$ and $Y$.

PROPOSITION P7. Let $\alpha, \beta$ be two nonempty isolated sets, $A = \text{Req}(\alpha)$ $B = \text{Req}(\beta)$ and

$$S = \{j(\xi \times \eta) \mid (\exists n)(n \geq 1 \text{ and } \xi = C(\alpha, n) \text{ and } \eta = C(\beta, n))\}.$$ 

If $\alpha$ and $\beta$ are regressive, i.e., $A, B \subseteq \Delta_R$ then $S$ is a gc-class with $\text{RET}(S) = \text{min}(A, B)$.

It can be shown that $S$ may be a gc-class while the sets $\alpha$ and $\beta$ are immune, but not both regressive.


DEFINITIONS. Let $p(x)$ be a partial recursive function and $S$ a gc-class. Then $p(x)$ is a gc-function of $S$, if
(α) $\sigma \subseteq \delta p$ and $p(\sigma) \in \xi(S)$,
(β) $(\forall x)[x \in \sigma \Rightarrow p(x) \in p(\sigma) \cdot \alpha]$, 
(γ) $\rho p \subseteq \delta p$ and $(\forall x)[x \in \delta p \Rightarrow p^2(x) = p(x)]$.

A gc-function is a partial recursive function which is a gc-function of at least one gc-class.

Every gc-class has at least one gc-function. For if a partial recursive function $p(x)$ is related to $S$ by (α) and (β), then $p(x)$ has a restriction which satisfies (α), (β) and (γ). With every partial recursive function $p(x)$ we associate the md-class $Gen(p) = \{p^{-1}(y) \mid y \in p p\}$ of r.e. sets. This md-class is empty if and only if $p(x)$ is nowhere defined.

**Proposition P8.** A partial recursive function $p(x)$ is a gc-function if and only if it satisfies (γ). Moreover, if $p(x)$ satisfies (γ), it is a gc-function of the class $S = \text{Gen}(p)$ with $\sigma = \delta p$ and $p(\sigma) = p p \in \xi(S)$.

**Proposition P9.** Let $p(x)$ be a gc-function of the gc-class $S$. Then

$$\delta p = \sigma \Leftrightarrow S = \text{Gen}(p).$$

**Definition I.** A class $S$ is primitive, if it satisfies one of the three conditions: (i) $S$ is empty, (ii) $S$ is a nonempty, finite md-class of r.e. sets, (iii) $S$ is a denumerable md-class of r.e. sets and there exists a recursive function $a(n, x)$ such that if $\alpha_n = pa(n, x)$, then $S$ consists of the distinct sets $\alpha_0, \alpha_1, \cdots$.

**Definition II.** A class $S$ is primitive, if it is a gc-class with a gc-function $p(x)$ such that $S = \text{Gen}(p)$.

**Definition III.** A class $S$ is primitive, if $S = \text{Gen}(p)$ for some partial recursive function $p(x)$.

**Proposition P10.** The three definitions of a primitive class are equivalent.

**Corollary.** A class $S$ is primitive if and only if it is a gc-class with a gc-function $p(x)$ such that $\delta p = \sigma$.

**Definition.** An md-class $T$ is a restriction of the gc-class $S$, if

(a) for every $\beta \in T$, there is an $\alpha_\beta$ such that $\beta \subseteq \alpha_\beta \subseteq S$,
(b) there is a $\gamma \in \xi(S)$ such that $\beta \in T \Rightarrow \gamma \cdot \alpha_\beta \subseteq \beta$.

**Proposition P11.** An md-class is a gc-class if and only if it is a restriction of some primitive gc-class.

While there are $c$ gc-classes, only $\aleph_0$ of them are primitive. For each RET $A$ there exists a gc-class with $A$ as its RET, but a primitive class can only have one of $0, 1, \cdots, R$ as its RET. The gc-sets of a primitive class $P$ are readily characterized. For if $P$ is finite, the gc-sets of $P$ are the choice sets of $P$, and if $P$ is infinite, say
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\[ P = (\alpha_0, \alpha_1, \cdots), \quad \alpha_n = pa(n, x), \]

\( a(n, x) \) a recursive function, then \( \gamma \in \xi(p) \) if and only if \( \gamma = pa(f_n, u_n) \), for a recursive permutation \( f_n \) and a recursive function \( u_n \). Finally, the restrictions of any given primitive class can be simply described. Thus Proposition P11 serves a purpose.

REFERENCES


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