RESEARCH ANNOUNCEMENTS

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MULTIPLIERS OF $p$-INTEGRABLE FUNCTIONS\textsuperscript{1}

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1. Introduction. Let $G$ be a locally compact Abelian group. Let $L^p(G)$ ($1 \leq p < \infty$) be the space of $p$-integrable functions (with respect to the Haar measure) with the usual norm. A multiplier of $L^p(G)$ is a bounded linear operator $T$ of $L^p(G)$ into $L^p(G)$ which commutes with the translation operators; that is, $\tau_y T = T \tau_y$ for all $y \in G$, where $\tau_y f(x) = f(x+y)$. The space of multipliers will be denoted by $M_p = M_p(G)$. It is known that $M_1$ is isomorphic and isometric to the space of bounded regular Baire measures on $G$ and that $M_2$ is isomorphic and isometric to $L^\infty(\Gamma)$, where $\Gamma$ is the character group of $G$, and thus $M_2$ is the conjugate space of the space $A(G)$ of continuous functions on $G$ which are Fourier transforms of elements of $L^1(\Gamma)$. Theorem 1 below asserts that, for $1 < p < \infty$, $M_p$ is also the conjugate space of a space $A_p$ of continuous functions on $G$. A corollary of this fact is that $M_p$ is the closure in the weak operator topology of the linear span of the translation operators. A theorem due to Hörmander relating tempered distributions on $R^n$ to $M_p(R^n)$ \cite{2}, is also an easy consequence of Theorem 1. In view of the fact that a multiplier $T$ can be identified with an element $T^\sim \in L^{\infty}(\Gamma)$ ($\Gamma$ being the character group of $G$), another consequence of Theorem 1 is that if $T \in M_p$, $T^\sim \ast \mu = U^\sim$ with $U \in M_p$, where $\mu$ is a bounded regular Baire measure on $\Gamma$. If $G$ is a noncommutative unimodular group, a proposition analogous to Theorem 1 holds for operators commuting with right (respec-

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tively, left) translations; the specialization to the case \( p = 2 \) yields results established by Segal in [7]. Theorem 7 relates lacunary subsets of a discrete Abelian group \( \Gamma \) to multipliers on \( L^p(G) \) where \( G \) is the character group of \( \Gamma \).

2. **The space of multipliers as a conjugate space.** Hereafter \( p \) will be a fixed real number, \( 1 < p < \infty \), and \( q \) will be such that \( 1/p + 1/q = 1 \). \( C_{00}(G) \) and \( C_0(G) \) will be, respectively, the space of continuous functions with compact support and the space of continuous functions vanishing at infinity on \( G \). The convolution with respect to the Haar measure on \( G \) is defined as \( f \ast g \) whenever it is well defined.

**Definition 1.** Let \( A_p \) be the space of functions \( h \in C_0(G) \) which can be written as \( h = \sum_{i=1}^{\infty} c_if_i \ast g_i \), where \( f_i \in L^p(G) \), \( g_i \in L^q(G) \) and \( \sum |c_i| \|f_i\|_p \|g_i\|_q < \infty \); for \( h \in A_p \) define

\[
\|h\|_{A_p} = \inf \left\{ \sum_{i=1}^{\infty} |c_i| \|f_i\|_p \|g_i\|_q : h = \sum_{i=1}^{\infty} c_if_i \ast g_i \right\}.
\]

**Theorem 1.** \( M_p \) is isometric and isomorphic to the conjugate space of \( A_p \), the element \( T \in M_p \) corresponding to the functional \( \phi_T(h) = \sum c_i(Tf_i \ast g_i)(0) \), where \( h = \sum c_if_i \ast g_i \). The weak operator topology of \( M_p \) coincides, on the unit sphere of \( M_p \), with the weak-star topology induced by \( A_p \).

One notices that if \( f, g \in C_{00}(G) \), \( T(f \ast g) = Tf \ast g \), so that \( T(f \ast g) \) is, after correction on a set of Haar measure zero, a continuous function. One can then define the functional \( \phi_T(h) = Th(0) \), on the space \( S \) of linear combinations of functions of the type \( f \ast g \) with \( f, g \in C_{00}(G) \).

Thus the proof of Theorem 1 consists essentially of showing that the completion of \( S \), under the norm

\[
\|h\| = \sup \{ |Th(0)| : T \in M_p, \|T\|_{M_p} \leq 1 \},
\]

is \( A_p \). This is accomplished using the fact that the \( BX \) topology on the conjugate space \( X^* \) of a Banach space \( X \) has the same continuous linear functionals as the \( X \) topology (weak star topology) (cf. [1, V, 5.6]).

**Corollary 2.** \( M_p \) is the closure, in the weak operator topology, of the span of the translation operators.

**Remark 3.** As a consequence of the Riesz convexity theorem [1, V, 10.11], and in view of the duality between \( L^p(G) \) and \( L^q(G) \), the restriction to \( C_{00}(G) \) of an element \( T \in M_p \) can be extended to an element of \( M_r \) for \( p \leq r \leq q \). Furthermore \( \|T\|_{M_p} = \|T\|_{M_q} \) and, if \( p \leq r \leq s \leq 2 \), \( \|T\|_{M_s} = \|T\|_{M_r} \). Dually one has \( A_p = A_q \) and, if \( p \leq r \leq s \leq 2 \), \( A_s \subseteq A_r \).
with \( || \cdot ||_{A_2} \leq || \cdot ||_{A_4} \). One should also notice that \( A_2(G) = A(G) \) is the space of Fourier transforms of elements of \( L^1(\Gamma) \), where \( \Gamma \) is the character group of \( G \). Thus, since \( A_2 \) is dense in \( A_p \), each \( T \in M_p \) corresponds biuniquely to an element \( T^{-\gamma} \) of \( L^{\infty}(\Gamma) \). \( T^{-\gamma} \) will be called the transform of \( T \).

**Remark 4.** A simple consequence of Theorem 1 and of the fact that \( A_2 = A(G) \) is continuously and densely embedded in \( A_p \), is the result proved by Hörmander in [2], stating that for \( G = \mathbb{R}^n \), \( M_p \) can be identified with a subspace of the space of tempered distributions on \( \mathbb{R}^n \). It suffices to notice that the space \( S \) of rapidly decreasing functions on \( \mathbb{R}^n \) is continuously and densely embedded in \( A(\mathbb{R}^n) \) and therefore in \( A_p \) (cf., e.g., [3, I, 1.7]).

**Corollary 5.** Let \( T \in M_p \) and let \( T^{-\gamma} \) be its transform in the sense of Remark 3; then, if \( \mu \) is a bounded regular Baire measure on \( \Gamma \) (the character group of \( G \)), \( T^{-\gamma} \ast \mu \) is also the transform of a multiplier in \( M_p \).

One shows that if \( \hat{\mu} \) is the Fourier-Stieltjes transform of \( \mu \), \( \hat{\mu}h \in A_p \) for every \( h \in A_p \); thus the functional \( \phi(h) = T(\hat{\mu}h)(0) \) on \( A_p \) defines a multiplier whose transform is \( T^{-\gamma} \ast \mu \).

**Remark 6.** Theorem 1 and Corollary 2 are valid for not necessarily commutative unimodular groups in the following sense: Let \( \mathfrak{L}_p \) (respectively, \( \mathfrak{R}_p \)) be the space of bounded linear operators on \( L^p(G) \) which commute with right (respectively, left) translations by elements of \( G \). Let \( A_p^1 \) (respectively, \( A_p^2 \)) be the space functions on \( G \) which can be written as \( \sum_{i=1}^{n} c_i f_i \ast g_i \) with \( f_i \in L^p(G), g_i \in L^q(G) \) (respectively, \( f_i \in L^q(G), g_i \in L^p(G) \)) with \( f, g, c_i \) satisfying the conditions of Definition 1 and with norms analogously defined; then \( \mathfrak{L}_p \) (respectively, \( \mathfrak{R}_p \)) is the conjugate space of \( A_p^1 \) (respectively, \( A_p^2 \)). Moreover, \( \mathfrak{L}_p \) (respectively, \( \mathfrak{R}_p \)) is the closure, in the weak operator topology, of the space of the left (respectively, right) translations. Thus the space \( \mathfrak{R}_p \) of the operators commuting with elements of \( \mathfrak{R}_p \) is \( \mathfrak{L}_p \) and conversely, so that \( \mathfrak{R}_p \cap \mathfrak{L}_p \) is the center of both \( \mathfrak{L}_p \) and \( \mathfrak{R}_p \). For the case \( p = 2 \) these results specialize to known results due to Segal [7]. One should also notice that \( A_p^1 = A_p^2 \) and therefore \( \mathfrak{L}_p \) is isometric and linearly isomorphic to \( \mathfrak{R}_p \).

### 3. Multipliers and lacunary sets.

Let \( G \) be a compact Abelian group, \( \Gamma \) its discrete character group. A set \( E \subseteq \Gamma \) is called a **Sidon set** (cf. [5, 5.7.2]) if every \( f \in C(G) \) with \( \hat{f}(\gamma) = 0 \) for \( \gamma \in E \) satisfies \( \sum |\hat{f}(\gamma)| < \infty \), or equivalently if for every bounded function \( \lambda(\gamma) \) on \( E \), there exists a Baire measure \( \mu \) on \( G \) such that \( \mu(\gamma) = \lambda(\gamma) \) for \( \gamma \in E \) (\( \hat{f} \) and \( \hat{\mu} \) denote, respectively, the Fourier transform and the Fourier-
Stieltjes transform of \( f \) and \( \mu \). As measures (operating by convolution) are exactly the operators on \( L^1(G) \) which commute with translations, it is natural to define an analogous concept for multipliers of \( L^p(G) \), recalling that each \( T \in M_p \) corresponds biuniquely to an element \( T^\sim \) of \( \mathcal{L}(\Gamma) \) (cf. Remark 3).

**Definition 2.** A set \( E \subseteq \Gamma \) is called a \( p \)-Sidon set if every \( f \in A_p \) such that \( \hat{f}(\gamma) = 0 \) for \( \gamma \in E \) satisfies
\[
\sum |\hat{f}(\gamma)| < \infty.
\]

**Theorem 7.** Let \( p \neq 2 \), then the following properties are equivalent for a subset \( E \) of \( \Gamma \):

(i) \( E \) is a \( p \)-Sidon set;

(ii) if \( \lambda \in L^\infty(\Gamma) \), there exists \( T \in M_p \) such that \( T^\sim(\gamma) = \lambda(\gamma) \) for \( \gamma \in E \);

(iii) if \( \lambda \in L^\infty(\Gamma) \), there exists \( T \in M_p \) satisfying (ii) and moreover such that \( T^\sim(\gamma) = 0 \) for \( \gamma \notin E \);

(iv) if \( f \in L^1(G) \) and \( \hat{f}(\gamma) = 0 \) for \( \gamma \in E \), then \( f \in L^r(G) \) where \( r = \max(p, q) \).

It should be noted that condition (iv) above is the defining property for what is called a lacunary set of order \( r \) or a \( \Lambda(r) \) set. Properties of these sets are investigated in [6] and [4, Chapter VIII]. In particular, it is known that a Sidon set is a lacunary set of order \( r \) for every \( r \) and hence a \( p \)-Sidon set for every \( p \). One should also notice that condition (iii) above implies that the characteristic function of a \( p \)-Sidon set is always the transform of a multiplier. The analogous statement for Sidon sets does not hold; indeed it is known that a Sidon set whose characteristic function is the Fourier-Stieltjes transform of a measure is necessarily finite.

**References**


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