1. Introduction. Let $U$ be the upper half plane. Let $\Sigma$ be the set of quasiconformal self-mappings of $U$ which leave 0, 1, and $\infty$ fixed. The universal Teichmüller space of Bers is the set $T$ of mappings $h: \mathbb{R} \to \mathbb{R}$ which are boundary values of mappings in $\Sigma$.

Let $M$ be the open unit ball in $L^\infty(U)$. For each $\mu$ in $M$, let $f^\mu$ be the unique mapping in $\Sigma$ which satisfies the Beltrami equation

$$ f^\mu = \mu f^\mu. $$

We map $M$ onto $T$ by sending $\mu$ to the boundary mapping of $f^\mu$. $T$ is given the quotient topology induced by the $L^\infty$ topology on $M$. The right translations, of the form $h \to h \circ h_0$, are homeomorphisms of $T$.

We shall also associate to each $\mu$ in $M$ a function $\phi^\mu$ holomorphic in the lower half plane $U^*$. For each $\mu$, let $w^\mu$ be the unique quasiconformal mapping of the plane on itself which is conformal in $U^*$, satisfies (1) in $U$, and leaves 0, 1, and $\infty$ fixed. $\phi^\mu$ is the Schwarzian derivative $\{w^\mu, z\}$ of $w^\mu$ in $U^*$. By Nehari [3], $\phi^\mu$ belongs to the Banach space $B$ of holomorphic functions $\psi$ on $U^*$ which satisfy

$$ \|\psi\| = \sup |(z - z^*)^2 \psi(z)| < \infty. $$

It is known [1, pp. 291–292] that $\phi^\mu = \phi^\nu$ if and only if $f^\mu$ and $f^\nu$ have the same boundary values. Hence, there is an injection $\theta: T \to B$ which sends the boundary function of $f^\mu$ to $\phi^\mu$. We shall write $\theta(T) = \Delta$.

Now let $G$ be a Fuchsian group on $U$; that is, a discontinuous group of conformal self-mappings of $U$, not necessarily finitely generated. The mapping $f$ in $\Sigma$ is compatible with $G$ if $f \circ A \circ f^{-1}$ is conformal for every $A$ in $G$. The Teichmüller space $T(G)$ is the set of $h$ in $T$ which are boundary values of mappings compatible with $G$. The space $B(G)$ of quadratic differentials is the set of $\phi$ in $B$ such that

$$ \phi(Az) A'(z)^2 = \phi(z) \quad \text{for all } A \text{ in } G. $$

Ahlfors proved in [1] that $\Delta$ is open in $B$. Bers [2] proved that $\theta$ maps $T$ homeomorphically on $\Delta$ and maps $T(G)$ onto an open subset of $B(G)$. These results are summed up in the following theorems:

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1 This research was supported by the National Science Foundation grant NSF-GP780.
Theorem 1. The mapping \( \mu \to \phi^\mu \) is continuous.

Theorem 2. The mapping \( \mu \to \phi^\mu \) is open.

Theorem 3. \( \theta(T(G)) \) is an open subset of \( B(G) \).

Our purpose here is to give new, more elementary proofs of Theorems 2 and 3. In particular, we notice that Theorem 3 is a straightforward consequence of Theorems 1 and 2 and the lemma in the next section.

2. The space \( D(G) \). For each Fuchsian group \( G \), we denote by \( D(G) \) the set of \( h \) in \( T \) such that \( h \circ A \circ h^{-1} \) is the boundary function of a conformal self-mapping of \( U \) for every \( A \) in \( G \). Clearly, \( T(G) \) is contained in \( D(G) \).

Lemma. \( \theta(D(G)) = B(G) \cap \Delta \).

Proof. For each \( A \) in \( G \) and \( \phi^\mu \) in \( \Delta \),

\[
\phi^\mu(Az) A'(z)^2 = \{ w^\mu, A z \} A'(z)^2 = \{ w^\mu \circ A, z \}.
\]

Therefore, \( \phi^\mu \in B(G) \cap \Delta \) if and only if for each \( A \) in \( G \), the restriction of \( w^\mu \circ A \circ (w^\mu)^{-1} \) to \( w^\mu(U^*\ast) \) is a linear transformation.

Let \( \phi^\mu \) belong to \( \theta(D(G)) \). Let \( f = f^\mu \) and \( w = w^\mu \). Let \( g \) be the conformal map of \( U \) onto \( w(U) \) such that \( w = g \circ f \). For each \( A \) in \( G \) there is a conformal map \( A_1 : U \to U \) which agrees with \( f \circ A \circ f^{-1} \) on the real axis. We put \( S \) equal to \( w \circ A \circ w^{-1} \) in \( w(U^\ast) \) and to \( g \circ A_1 \circ g^{-1} \) in the closure of \( w(U) \). \( S \) is quasiconformal everywhere and conformal off \( w(R) \). Hence \( S \) is everywhere conformal, and \( \phi^\mu \in B(G) \cap \Delta \).

Conversely, suppose \( \phi^\mu \in B(G) \cap \Delta \). Let \( w = w^\mu \), \( f = f^\mu \), and \( g = w \circ f^{-1} \). Given \( A \) in \( G \), let \( S \) be the linear transformation which agrees with \( w \circ A \circ w^{-1} \) in \( w(U^*\ast) \). By continuity, \( S \circ w = w \circ A \) on the real axis. Therefore, \( f \circ A \circ f^{-1} = g^{-1} \circ S \circ g \) on \( R \), and the boundary function \( h \) of \( f \) belongs to \( D(G) \). But \( \theta(h) = \phi^\mu \). Q.E.D.

3. Proof of Theorem 2. Let \( \phi_0 = \phi^\mu \) be a point of \( \Delta \). We must show that every neighborhood of \( \mu \) covers a neighborhood of \( \phi_0 \). Ahlfors [1] proves that if \( \| \phi - \phi_0 \| \) is sufficiently small, \( \phi \) belongs to \( \Delta \). With Ahlfors, we write \( \phi = \{ w^\ast, z \} \) where \( w^\ast = f \circ w^\mu \). It suffices to prove that the complex dilatation of \( f \) tends to zero with \( \| \phi - \phi_0 \| \).

According to [1, p. 300], \( f \) is the limit of a sequence of mappings \( f_n \). Formula (13) of [1] and the chain rule, we compute that the complex dilatation \( \rho_n \) of \( f_n \) satisfies

\[
\| \rho_n \|_\infty < \frac{\| \phi - \phi_0 \|}{\delta - \| \phi - \phi_0 \|}.
\]
where \( \delta \) is a positive constant depending only on \( \mu \). Obviously, \( \| \rho_n \|_\infty \) tends to zero with \( \| \phi - \phi_0 \| \). Q.E.D.

4. **Proof of Theorem 3.** We show first that \( \theta(T(G)) \) contains a neighborhood of the origin in \( B(G) \). It is well known [1, pp. 297–299] that every \( \phi \) in \( B \) with \( \| \phi \| < 2 \) has the form \( \phi^* \) for
\[
\mu(z) = \frac{1}{2}(z - z^*)^2 \phi(z^*).
\]
Moreover, it is a simple consequence of the chain rule that \( \phi^* \) is compatible with \( G \) if and only if
\[
\mu(Az) = \mu(z) A'(z) / A'(z)^* \quad \text{for all } A \text{ in } G.
\]
If \( \phi \in B(G) \) and \( \| \phi \| < 2 \), the \( \mu \) in (2) satisfies (3). Hence, \( \theta(T(G)) \) contains the open unit ball in \( B(G) \).

Now let \( f^* \) be any mapping compatible with \( G \) and let \( G^* \) be the Fuchsian group \( f^* \circ G \circ (f^*)^{-1} \). Let \( \alpha: T \to T \) be the right translation which carries the boundary mapping of \( f^* \) to the identity. It is obvious that \( \alpha \) maps \( T(G) \) onto \( T(G^*) \) and \( D(G) \) onto \( D(G^*) \). Let \( \beta: \Delta \to \Delta \) be the homeomorphism \( \theta \circ \alpha \circ \theta^{-1} \). By the Lemma, \( \beta \) maps the open set \( B(G) \cap \Delta \) in \( B(G) \) onto the open set \( B(G^*) \cap \Delta \) in \( B(G^*) \). Moreover, \( \beta \) maps \( \phi^* \) to zero.

We have seen that \( \theta(T(G^*)) \) contains the open unit ball \( N \) in \( B(G^*) \). Since \( \alpha \) maps \( T(G) \) on \( T(G^*) \), \( \beta^{-1}(N) \) is contained in \( \theta(T(G)) \). Since \( \beta \) is a homeomorphism of \( B(G) \cap \Delta \) on \( B(G^*) \cap \Delta \), \( \beta^{-1}(N) \) is open in \( B(G) \). Therefore, \( \theta(T(G)) \) contains a neighborhood of \( \phi^* \) in \( B(G) \). Since \( f^* \) was any mapping compatible with \( G \), \( \theta(T(G)) \) is an open set. Q.E.D.

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