THE EQUIVALENCE OF THE ANNULUS CONJECTURE
AND THE SLAB CONJECTURE

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In [1], the author showed that the Slab Conjecture implies the Annulus Conjecture.

The purpose of this paper is to show that the Annulus Conjecture implies the Slab Conjecture for \( n > 3 \) and hence the two conjectures are equivalent for \( n > 3 \).

\( R^n, S^n \) will denote \( n \)-space and the \( n \)-sphere, respectively. A \( k \)-manifold \( N \) is embedded in a locally flat manner in an \( n \)-manifold \( M \) provided each point of \( N \) has a neighborhood \( U \) in \( M \) such that \( (U, U \cap N) \approx (R^n, R^k) \).

**The Annulus Conjecture.** Let \( S^1 _1, S^2 _1 \) be disjoint locally flat \( (n-1) \)-spheres embedded in \( S^n \) and let \( M \) be the submanifold of \( S^n \) bounded by \( S^1 _1 \cup S^2 _1 \). Then \( M \approx S^{n-1} \times [0, 1] \).

**The Slab Conjecture.** Let \( R^1 _1, R^2 _1 \) be disjoint locally flat \( n-1 \) spaces embedded as closed subsets of \( R^n \) and let \( M \) be the submanifold of \( R^n \) bounded by \( R^1 _1 \cup R^2 _1 \). Then \( M \approx R^{n-1} \times [0, 1] \).

**Theorem.** The Annulus Conjecture implies the Slab Conjecture for \( n > 3 \).

**Proof.** Let \( R^1 _1, R^2 _1 \) be disjoint locally flat \( n-1 \) spaces embedded as closed subsets of \( R^n \), \( n > 3 \), and let \( M \) be the submanifold of \( R^n \) bounded by \( R^1 _1 \cup R^2 _1 \). Let \( S^n = R^n \cup \{p\} \) be the one-point compactification of \( R^n \) and \( S^i _1 = R^i _1 \cup \{p\} \) for \( i = 1, 2 \). By the corollary to Theorem 2 of [2], \( S^i _1 \) is flat for \( i = 1, 2 \). Hence, we may assume that \( S^1 _1 = S^n \), that \( S^2 _1 \) lies in the northern hemisphere of \( S^n \), and that \( S^1 _1 \cap S^2 _1 = \{p\} \).

Let \( B^{n-1} \) be the unit ball in \( S^i _1 = S^{n-1} \) with center \( p \), \( r \) the south pole of \( S^n \), \( q \) the midpoint of the line segment joining \( p \) to \( r \) in \( S^n \), \( L \) the line segment joining \( p \) to \( q \) in \( S^n \), and \( B^r, B^q \) the cones \( (n\)-balls) in \( S^n \) with bases \( B^{n-1} \) and cone points \( r, q \) respectively. (See Figure 1.) Now, let \( S^i _2 = [S^i _1 \cup B^i _r] - \text{Int}(B^{n-1}) \). Then \( S^i _2 \) is a flat \( n-1 \) sphere in \( S^n \) and \( S^1 _2 \cap S^2 _2 = \emptyset \). By the Annulus Conjecture, \( M \cup B^r \approx A^n \) is an \( n \)-annulus. We will complete the proof by showing that \( M \cup \{p\} \) is homeomorphic to the decomposition space \( A^n/L \) and applying Lemma 3 of [3].
By Theorem II.3 of [1], \( M_2 = M - R_{n-1} \approx R_{n-1} \times [0, 1) \) under some homeomorphism \( h \). Take \( T = h^{-1} \left[ h(B_{n-1} - p) \times [0, \frac{1}{2}] \right], T_r = T \cup B^n_r, \) and \( T_q = T \cup B^n_q \). Then \( T_r, T_q \) are \( n \)-balls with \( T_q \subseteq T_r \).

There is a natural map \( f \) of \( T_r \) onto itself such that the following hold:

1. \( f | T_r = 1 \),
2. \( f | T_r - L \) is a homeomorphism,
3. \( f(L) = p \),
4. \( f(\text{CL}(B^n_r - B^n_q)) = B^n_q \).

\( f \) is obtained by pushing \( B^n_q \) up into \( T \cup \{ p \} \) making use of the parameterization induced on \( T \) by \( h^{-1} \). \( f \) extends to a map of \( S^n \) onto itself by \( f | S^n - T_r = 1 \).

Since \( f(A^n) = M \cup \{ p \}, f | A^n - L \) is a homeomorphism and \( f(L) = p \), it follows that \( M \cup \{ p \} \approx A^n / L \). By Lemma 3 of [3], since \( L \) is a flat arc in \( A^n \) with endpoints \( p \in S_{n-1}^2, q \in S_{n-1}^3 \) and \( L - (p \cup q) \subseteq \text{Int} A^n \), \( A^n / L \) is a pinched annulus, that is, \( A^n / L \) is homeomorphic to the one-point compactification of \( R_{n-1} \times [0, 1] \). Thus \( M = R_{n-1} \times [0, 1] \) and the theorem is proved.

**Corollary.** The Annulus Conjecture is equivalent to the Slab Conjecture for \( n > 3 \).

**References**

3. ———, *Some relations between the Annulus Conjecture and union of flat cells theorems* (to appear).

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