\[ \| F \|_{\mathcal{F}_p} \leq 2K_p N^{1/q - 1} \gamma^{1/p - 1} = 2K_p \varepsilon^{1/q - 1} \gamma^{1/p - 1} \]
so that, if \( N \) is large enough \( \| F \|_{\mathcal{F}_p} < \varepsilon \). Put
\[ E_{\varepsilon,p} = \{ x ; F(x) = 1 \} . \]
Now \( F(x) = 1 \Leftrightarrow f_\gamma (\lambda, x) = 1 \) for all \( j \), so that measure \( (E_{\varepsilon,p}) \geq 2\pi - N \gamma = 2\pi - \varepsilon . \)

Let \( \mu \) be carried by \( E_{\varepsilon,p} \), \( \mu \subset \mathcal{F}_p \), then
\[ | \hat{\mu} (n) | = | \int e^{-inx} d\mu | = | \sum \hat{F} (m-n) \hat{\mu} (m) | \leq \| \hat{F} \|_{\mathcal{F}_p} \| \mu \|_{\mathcal{F}_p} < \varepsilon | \mu |_{\mathcal{F}_p} \]
which proves the lemma.

**Theorem.** There exists a set \( E \) of positive measure on \( T \) which is a set of uniqueness for \( \bigcup_{p<2} \mathcal{L}^p \).

**Proof.** Take \( \varepsilon_n = 10^{-n} \), \( p_n = 2 - \varepsilon_n \),
\[ E = \cap_{n=36}^{\infty} E_{\varepsilon_n, p_n} . \]

**Yale University**

---

**INJECTIVE ENVELOPES OF BANACH SPACES**

**BY HENRY B. COHEN**

Communicated by V. Klee, July 30, 1964

**1. Introduction.** We consider the category whose objects are Banach spaces and whose maps are the linear operators of norm not exceeding 1 from one Banach space into another. A Banach space \( Z \) is injective if it has the same Hahn-Banach extension property that is possessed by the scalars (real or complex); that is, any \( Z \)-valued map from a subspace of a Banach space \( Y \) extends to a \( Z \)-valued map of the same norm on all of \( Y \). An injective envelope of a Banach space \( B \) is a pair \( (I, \varepsilon B) \), \( \varepsilon B \) an injective Banach space and \( I : B \rightarrow \varepsilon B \) a linear isometry (our linear isometries need not be onto), such that the only subspace of \( \varepsilon B \) that is injective and contains \( I[B] \) is \( \varepsilon B \) itself. In this note, we demonstrate the existence and uniqueness of the injective envelope of a Banach space and, in the process, we give a short proof of the fact that an injective Banach space is linearly isometric with a function space \( C(M) \), \( M \) compact Hausdorff and extremally disconnected.
For $X$ a compact Hausdorff space, $C(X)$ denotes the Banach space of all continuous scalar-valued functions on $X$ with the sup norm.

A topological space is \textit{extremally disconnected} if the closure of every open subset is open; a continuous function is \textit{minimal} if it is onto, but no longer onto when restricted to an arbitrary closed proper subset of its domain. Fundamental to our construction is Gleason’s result [1]: for any compact Hausdorff space $X$, there is a minimal continuous function $i: M \to X$ with $M$ compact Hausdorff and extremally disconnected. The writer wishes to thank J. Isbell for suggesting the problem of determining the injective envelope of a Banach space; Isbell conjectured that the injective envelope of a function space $C(X)$ would be $(I, C(M))$ where $I$ is the linear isometry induced by Gleason’s function $i: M \to X$.

**Theorem 1 (Nachbin-Goodner-Hasumi).** \textit{If $M$ is compact Hausdorff and extremally disconnected, then $C(M)$ is injective.}

**Proof.** Phillips [6] showed that the Banach space $m(D)$ of bounded scalar-valued functions on a set $D$ is injective; consequently, a function space $C(K)$, $K$ the Stone-Cech compactification of a discrete space, is injective. Given a compact Hausdorff extremally disconnected space $M$, the combined results of Gleason [1] and Rainwater [7] imply that $M$ is a retract of a suitably chosen Stone-Cech compactification $K$ of a discrete space. It follows that $C(M)$ is a retract of $C(K)$ and therefore injective.

2. **Construction of the envelope.** For any subset $Q$ of a real or complex linear space, $EQ$ denotes the set of extremal points of $Q$ and $coQ$ the convex hull. For any Banach space $B$, $B^*$ denotes the adjoint space and $B^{**}$ the unit sphere of $B^*$; $cl^*Q$ denotes the weak* closure of a subset $Q$ of $B^*$. The set of all scalars of norm 1 is denoted by $C_0$.

Let $B$ be a Banach space. Let $U$ and $X$ be subsets of $B^*$ satisfying

1. $X$ is weak* closed; $U$ is contained in $EB^* \cap X$.
2. $cl^*U = X$.
3. $cl^*C_0U = cl^*EB^*$. (4) For each $u$ in $U$, $C_0u \cap X = \{u\}$. For $B$ real, such a $U$ and $X$ are easily constructed as follows. Let $W$ be a subset of $cl^*EB^*$ maximal with respect to being both open in $cl^*EB^*$ and disjoint from $-W$. Then $W \cup -W$ is weak* dense in $cl^*EB^*$; hence, so is $(EB^* \cap W) \cup -(EB^* \cap W)$. Let $U = EB^* \cap W$ and $X = cl^*U$.

For $B$ complex, proceed as follows. Call a subset $W$ of $B^*$ deleted if $w \in W$ implies $kw \in W$ for all but exactly one $k$ in $C_0$, and say that $W$ is \textit{circled} if $C_0W \subseteq W$. If $W$ is circled, so is $cl^*W$; in particular, $cl^*EB^*$ is circled. Every nonvoid open circled subset $W$ of $cl^*EB^*$ contains a nonvoid open deleted subset. For choose $w$ in $W$ and $b$ in $B$ such
that \( w(b) \subseteq D \), the open unit disc in the plane with the interval \([0, 1)\) removed. If \( J(b) \) is the weak* continuous functional determined by \( b \), i.e., \( J(b)(z) = z(b) \) for all \( z \) in \( B^* \), then \( J(b)^{-1}(D) \cap W \) is the required set. Using Zorn's lemma, let \( W \) be a subset of \( \text{cl}^*EB^* \) maximal with respect to being both open in \( \text{cl}^*EB^* \) and deleted. Then \( C_0W \) is dense in \( \text{cl}^*EB^* \). Since \( C_0W \) is open and dense in \( \text{cl}^*EB^* \), \( EB^* \cap C_0W \) is dense there, too. Let \( U \) denote the set of all \( u \) in \( EB^* \) such that \( ku \subseteq W \) for all \( k \) in \( C_0 \), \( k \neq 1 \). Define \( X = \text{cl}^*U \). Thus (1) and (2) are satisfied trivially, and (3) follows from the equality \( C_0U = EB^* \cap C_0W \). The fact that \( W \) and \( X \) are disjoint yields (4).

With \( U \) and \( X \) thus chosen, \( X \) is a compact Hausdorff space; let \( i: M \rightarrow X \) be Gleason's function. Let \( I: B \rightarrow C(M) \) be defined by \( I(b)(m) = i(m)(b) \). The continuity of \( I \) is used to show that \( I \) is well defined, \( I \) is obviously linear, and \( I \) is a map because \( X \) is contained in \( B^* \). To show \( I \) is an isometry, let \( b \subseteq B \) and choose \( v \in EB^* \) such that \( \|v\| = |v(b)| \). Using (3), let \( v(a) \) be a net in \( C_0U \) converging to \( v \) in the weak* topology; hence, \( |v(a)(b)| \) converges to \( \|b\| \). Given \( \epsilon > 0 \), choose \( \alpha \) such that \( \|b\| - \epsilon < |v(a)(b)| \). For a unique \( k \) in \( C_0 \), \( v(a) = ku \) in \( U \). Thus \( \|b\| - \epsilon < \|u(b)\| = |i(m')(b)| = |I(b)(m')| \leq \|I(b)\| \) for a suitable \( m' \in M \). Since \( \epsilon > 0 \) was arbitrary, \( \|b\| \leq \|I(b)\| \).

**Theorem 2.** The pair \( (I, C(M)) \) is the essentially unique injective envelope of \( B \); indeed, given a Banach space \( Y \), a linear isometry \( G: B \rightarrow Y \), and any map \( H: C(M) \rightarrow Y \) such that \( H \circ I = G \), then \( H \) is a linear isometry.

**Proof.** We first prove \( H \) is a linear isometry; note, \( \|H\| \leq 1 \) by assumption. Let \( f \) belong to \( C(M) \) and \( \epsilon > 0 \) be given. Assume that \( f \) takes on the value \( \|f\| \), no restriction since we are interested in the norm of \( f \) and can therefore work with \( kf \) for any \( k \) in \( C_0 \). Set \( M(\epsilon) \) equal to the set of all \( m \) in \( M \) such that \( f(m) \) lies in \( D(\epsilon) \), the open disc in the plane of radius \( \epsilon \) centered at \( \|f\| \). Because \( i \) is minimal, \( i[M \setminus M(\epsilon)] \) is a proper closed subset of \( X \). By the fact (2) that \( U \) is dense in \( X \), let \( u \) belong to \( U \) and to \( X \setminus i[M \setminus M(\epsilon)] \); consequently, \( i^{-1}(u) \subseteq M(\epsilon) \). Letting \( e: M \rightarrow C(M)^* \) denote the homeomorphism \( e(m)(g) = g(m) \), \( e[i^{-1}(u)] \) is a collection of functionals whose values at \( f \) lie in \( D(\epsilon) \). Consequently,

\[
\text{if } z \text{ is in cl}^*e[i^{-1}(u)], \quad \text{then } \|f\| - \epsilon \leq |z(f)|.
\]

Next, suppose \( z \) is in \( EC(M)^* \cap I^{*^{-1}}(u) \). Then \( z = ke(m) \) for some \( m \) in \( M \), \( k \) in \( C_0 \); and so, \( u = I^*z = kI^*e(m) = ki(m) \). Therefore \( i(m) \) is a member of \( C_0u \cap X \) so by (4), \( i(m) = u \) and \( k = 1 \). Therefore \( m \) is in \( i^{-1}(u) \) and \( z = e(m) \) is in \( e[i^{-1}(u)] \). Taking closed convex hulls:
The Hahn-Banach theorem applied to $u \circ G^{-1}$ yields a $y$ in $Y$ such that $G^*y = u$; hence, $I^*(H^*y) = (H \circ I)^*y = G^*y = u$. Consequently $H^*y$ is in $C(M)^* \cap I^{*-1}(u)$ a set contained in $cl^{*co}(EC(M)^* \cap I^{*-1}(u))$ (take the weak* closed convex hull of both sides of $E(C(M)^* \cap I^{*-1}(u)) \subseteq EC(M)^* \cap I^{*-1}(u)$, reducing the left-hand side of the resulting inclusion by means of the Krein-Milman theorem). Using (b) and then (a), $||f|| - \epsilon \leq |H^*y(f)| = |y(H(f))| \leq ||H(f)||$. Since $\epsilon$ was arbitrary, $||f|| \leq ||H(f)||$. Therefore, $H$ is an isometry.

Suppose $Z$ is a subspace of $C(M)$ containing $I[B]$. If $Z$ is injective, there is a map $H: C(M) \to Z$ such that $H(z) = z$ for all $z$ in $Z$. Letting $G: B \to Z$ denote the map $I$ with range restricted, $H \circ I = G$ so, by the above, $H$ is an isometry. But the only way that $H$ can be 1—1 is for $Z$ to be all of $C(M)$. Therefore, $(I, C(M))$ is an injective envelope of $B$.

This injective envelope is unique in the sense that if $(G, Y)$ is another injective envelope of $B$, there is a linear isometry $H: C(M) \to Y$ onto such that $H \circ I = G$. For $Y$ injective provides a map $H: C(M) \to Y$ such that $H \circ I = G$, and the lemma implies that $H$ is an isometry. This means that $H[C(M)]$ is an injective Banach space which, as a subspace of $Y$, contains $G[B]$. Therefore, $H[C(M)] = Y$; i.e., $H$ is onto.

**Theorem 3 (Kelley-Hasumi).** An injective Banach space $B$ is linearly isometric with a function space $C(M)$, $M$ compact Hausdorff and extremally disconnected.

**Proof.** Suppose $B$ is injective. Then if $I: B \to C(M)$ is the map constructed above, $I[B]$ is injective and therefore equal to $C(M)$.

**References**


University of Pittsburgh