If $\mathcal{H}$ is a separable (complex) Hilbert space, and $A$ is a (bounded, linear) operator on $\mathcal{H}$, then $A$ is a commutator if there exist operators $B$ and $C$ on $\mathcal{H}$ such that $A = BC - CB$. It was shown by Wintner [8] and also by Wielandt [7] that no nonzero scalar multiple of the identity operator $I$ on $\mathcal{H}$ is a commutator, and this was improved by Halmos [5] who showed that no operator of the form $\lambda I + C$ is a commutator, where $\lambda \neq 0$ and $C$ is a compact operator. The purpose of this note is to announce the following theorem and give some indication of its proof. Details of the results described below will appear elsewhere [2].

**Theorem.** An operator $A$ on a separable Hilbert space $\mathcal{H}$ is a commutator if and only if $A$ is not of the form $\lambda I + C$ where $\lambda \neq 0$ and $C$ is a compact operator.

This theorem furnishes the solution to several problems concerning commutators posed by Halmos in [4] and [5]. In particular it is interesting to note that the identity operator is the limit in the norm of commutators and that there exists a commutator whose spectrum consists of the number 1 alone.

**Indication of the Proof.** We must show that every operator that is not of the form $\lambda I + C$, with $\lambda \neq 0$ and $C$ compact, is a commutator. These operators fall naturally into two classes; viz., the class of compact operators, which was shown to consist entirely of commutators in [1], and the class consisting of all operators that cannot be written in the form $\lambda I + C$ for any scalar $\lambda$ (0 or not) and compact $C$. We denote this latter class by $(F)$, and the first problem is to obtain a more usable characterization of the operators of this class. To this end we define for an arbitrary operator $T$ on $\mathcal{H}$ the function

$$\eta_T(x) = \|Tx - (Tx, x)x\|, \quad x \in \mathcal{H}, \|x\| = 1,$$

and denote by $\eta_T(\mathcal{M})$ the supremum over the subspace $\mathcal{M} \subseteq \mathcal{H}$ of $\eta_T(x)$.

**Proposition 1.** An operator $T$ is of type $(F)$ if and only if $\inf \eta_T(\mathcal{M}) > 0$ where the infimum is taken over all cofinite-dimensional subspaces $\mathcal{M}$ of $\mathcal{H}$. 

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This proposition may then be employed to yield a "standard form" for operators of type $(F)$.

**Proposition 2.** Every operator of type $(F)$ is similar to an operator of the form

$$
\begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} & I \\
A_{31} & A_{32} & 0
\end{bmatrix}
$$

acting in the usual fashion on a Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$. (The $A_{ij}$ are, of course, operators on $\mathcal{H}$.)

It is easily seen from this that to complete the proof it suffices to show that every $2 \times 2$ operator matrix of the form

$$
\begin{pmatrix}
A & U \\
B & 0
\end{pmatrix},
$$

where $U$ is an isometry with infinite deficiency, is a commutator. This is accomplished by making a fairly intricate sequence of computations involving $2 \times 2$ matrices with operator entries. A central tool used in this argument is the result [6] that every operator with an infinite-dimensional null space is a commutator.

We note in conclusion that the restriction to separable spaces in the statement of the above theorem is for the sake of simplicity only; analogous results hold for an arbitrary infinite-dimensional Hilbert space.

**Bibliography**


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