ORDINARY MEANS IMPLY RECURRENT MEANS

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Introduction. Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space, let \(T\) be a positive linear operator from \(L_1(X)\) to \(L_1(X)\) whose norm is less than or equal to one. Let \(\{w_k\}, k \geq 1\), be a sequence of non-negative numbers whose sum is one and let \(\{u_k\}, k \geq 0\), be the sequence defined by \(u_n = w_1 u_{n-1} + \cdots + w_n u_0, u_0 = 1\). Set, for any pair of functions \(f\) in \(L_1(X)\) and \(p\) in \(L_1(X)\), \(p \geq 0\), \(Q_n(f, p) = Z_n(f)/Z_n(p)\), \(Z_n(g) = \sum_0^{n-1} u_k T^k g\). Baxter, \([2]\), \([3]\) utilizing \([8]\), has obtained the following result:

**Theorem 1.** The ratios \(Q_n(f, p)\) have a finite limit almost everywhere on the set where \(p>0\).

The method of proof given by Baxter is a considerable and non-trivial application of the methods given in \([4]\). The theorem reduces to that of \([4]\) if one takes \(w_1 = 1, w_k = 0, k \geq 2\). The purpose of the present note is to show that the theorem of \([4]\) yields Theorem 1 directly and in a stronger form. The stronger form of Theorem 1 gives convergence almost everywhere on the set where \(\sum_0^\infty u_k T^k p > 0\) and answers a question raised in \([3]\). Our proof is also sufficient to yield the theorem of \([1]\) (see \([7]\)).

1. Proof. Let \((I, \mathcal{A}, m)\) be the measure space obtained by taking \(I\) to be the positive integers, \(\mathcal{A}\) the Borel field of all subsets of \(I\), and \(m\) the measure given by \(m(\{1\}) = 1\) and, for \(i \geq 2\), by

\[
m(\{i\}) = 1 - w_1 - \cdots - w_{i-1}, \quad \beta_n = w_n/(1 - w_1 - \cdots - w_{n-1}),
\]

\(n \geq 2, \beta_1 = w_1\).

Let \(P\) be the transformation of \(L_1(I)\) to \(L_1(I)\) defined by left multiplication by the matrix

\[
P = \begin{pmatrix}
\beta_1 & 1 - \beta_1 & 0 & 0 & \cdots \\
\beta_2 & 0 & 1 - \beta_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}.
\]

We use \(P\) to denote the transformation and the matrix and represent the elements of \(L_1(I)\) as column vectors. It follows easily that \(\|P\| = 1\),
that \( P \) is positive and, setting \( P^n = (p_{ij}(n)) \), that \( u_n = p_{11}(n), n \geq 0. \)

Taking \( (Y, \mathfrak{F}, \gamma) \) to be the direct product of \((I, \mathfrak{B}, m)\) and \((X, \mathfrak{B}, \mu)\) and \( U \) the direct product of \( P \) and \( T \), it follows that \( U \) is a positive linear operator from \( L_1(Y) \) to \( L_1(Y) \) and that the norm of \( U \) is less than or equal to one. We may therefore apply the ratio theorem of [4] to \( U \) with \( f(y) = f(i, x) = \delta_{i1} \cdot f(x), \quad \tilde{p}(y) = p(i, x) = \delta_{i1} \cdot p(x) \) to obtain Theorem 1 with convergence almost everywhere on the set where \( \sum_{k=0}^\infty u_k T^k p > 0 \), since \( U^k f(i, x) = p_{11}(k) T^k f(x), \quad U^k p(i, x) = p_{11}(k) T^k p(x) \) and \( p_{11}(k) = u_k \).

**BIBLIOGRAPHY**


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1 This is related to renewal theory. See [5] for a discussion of relevant facts.