THE DIRICHLET PROBLEM FOR HOMOGENEOUS
ELLiptIC OPERATORS IN A HALF SPACE

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In a recent series of papers, Lions and Magenes [4], [5] study ex­
tensively boundary-value problems for elliptic operators. A proto­
type of problem that cannot be attacked by their methods is the
Dirichlet problem for powers of the Laplacian in a half space. By
using the completion of the space of smooth functions with respect
to the Dirichlet norm (§1), we are able to solve this problem.

We obtain, then, for a general class of homogeneous elliptic oper­
ators defined in a half space, isomorphism theorems establishing
existence and uniqueness for the Dirichlet problem (§§2, 4 and 5),
a regularity result (§3) and trace theorems (§§2, 3 and 4). Using
the theory of interpolation [3] we obtain other isomorphism theo­
rems between general classes of interpolated spaces. Among these,
we can characterize the spaces of boundary values (Theorem 5.2).

Proofs will appear elsewhere. The writer is indebted to J.-L. Lions
for suggestions and criticism.

1. Preliminaries. Let \( \mathbb{R}^n \) be the Euclidean space of \( n \) dimensions,
its elements being denoted by \( x = (x_1, \cdots, x_n) \). We denote by \( K_+^n \)
(resp. \( K_+^n \)) the set of elements \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n \) such that
\( x_n > 0 \) (resp. \( x_n \geq 0 \)). If \( p = (p_1, \cdots, p_n) \) is an \( n \)-tuple of integers \( \geq 0 \),
let \( D^p = D_{p_1}^1, \cdots, D_{p_n}^n \), where \( D_j = (1/\partial) (\partial/\partial x_j), \ 1 \leq j \leq n \), and let
\( |p| = p_1 + \cdots + p_n \) be the order of \( D^p \). If \( \Omega \) is an open subset of
\( \mathbb{R}^n \), we denote by \( C_c^\infty(\Omega) \) the space of infinitely differentiable functions
with compact support in \( \Omega \).

**Definition 1.1.** We denote by \( \mathcal{D}^m(\mathbb{R}^n) \) the completion of \( C_c^\infty(\mathbb{R}^n) \), with
respect to the following norm:

\[
\| \phi \|_{\mathcal{D}^m(\mathbb{R}^n)} = \left( \sum_{|p|=m} \| D^p \phi \|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2}.
\]

Clearly, \( \mathcal{D}^m(\mathbb{R}^n) \) is a Hilbert space. If \( n > 2m \) (an assumption that we
shall make throughout this paper), we have the Sobolev inequality
[6]:

\[
\| \phi \|_{L^4(\mathbb{R}^n)} \leq c \| \phi \|_{\mathcal{D}^m(\mathbb{R}^n)},
\]

for all \( \phi \in C_c^\infty(\mathbb{R}^n) \),

where

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and $c$ is a constant independent of $\phi$. It follows that $D^m(\mathbb{R}^n) \subset L^q(m)(\mathbb{R}^n)$, the imbedding being continuous. In fact, we have a more precise characterization of the elements of $D^m(\mathbb{R}^n)$.

**Theorem 1.1.** The space $D^m(\mathbb{R}^n)$ can be identified in the algebraic and topological senses to the space:

\[(1.3) \{ u \in L^q(m)(\mathbb{R}^n) : D^pu \in L^q(m-j)(\mathbb{R}^n), \quad |p| = j, \quad 0 \leq j \leq m \}, \]

where

\[(1.4) \frac{1}{q(m-j)} = \frac{1}{2} - \frac{m-j}{n}, \quad 0 \leq j \leq m, \]

equipped with the norm

\[ \sum_{|p| = j = 0}^m \| D^pu \|_{L^q(m-j)(\mathbb{R}^n)}. \]

In order to study the Dirichlet problem for homogeneous differential operators of order $2m$ in the half space $\mathbb{R}^n_+$, we introduce the following

**Definition 1.2.** We denote by $D^m(\mathbb{R}^n_+)$ the space

\[ D^m(\mathbb{R}^n_+) = \{ u \in L^q(m)(\mathbb{R}^n_+) : D^pu \in L^q(m-j)(\mathbb{R}^n_+), \quad |p| = j, \quad 0 \leq j \leq m \}, \]

where $q(m-j)$ is given by (1.4), equipped with the norm

\[(1.5) \| u \|_{D^m(\mathbb{R}^n_+)} = \sum_{|p| = j = 0}^m \| D^pu \|_{L^q(m-j)(\mathbb{R}^n_+)}.

Also let $D^m_0(\mathbb{R}^n_+)$ be the closure of $C^\infty_0(\mathbb{R}^n_+)$ in $D^m(\mathbb{R}^n_+)$ and let $D^{-m}(\mathbb{R}^n_+)$ be the dual of $D^m_0(\mathbb{R}^n_+)$.\n
2. **Isomorphism theorems.** Consider now the integro-differential operator defined on $D^m(\mathbb{R}^n_+) \times D^m(\mathbb{R}^n_+)$,

\[(2.1) a(u, v) = \sum_{|p| = |q| = m} \int_{\mathbb{R}^n_+} a_{pq}(x) D^pu D^qv dx, \]

where we assume, for simplicity, that the $a_{pq}(x)$ are $C^\infty$ functions on $\mathbb{R}^n_+$ and uniformly bounded. Assume that $a(u, v)$ satisfies the ellipticity condition:
Then one can prove the following (Lax-Milgram lemma [2]).

**Theorem 2.1.** Given an element \( f \in D^{-m}(R^+_n) \) there is a unique \( u \in D^m_0(R^+_n) \) such that

\[
(2.3) \quad a(u, v) = \langle f, v \rangle \quad \text{for all } v \in D^m_0(R^+_n),
\]

where \( \langle , \rangle \) denotes the pairing between \( D^m_0(R^+_n) \) and \( D^{-m}(R^+_n) \).

If

\[
(2.4) \quad Au = \sum_{|p| = |q| = m} (-1)^m a_p(x) D^q u
\]

is the differential operator associated to (2.1), we derive the

**Corollary 2.1.** \( A \) is an isomorphism of \( D^m_0(R^+_n) \) onto \( D^{-m}(R^+_n) \).

In other words, the homogeneous Dirichlet problem for \( A \) has a unique solution in \( D^m_0(R^+_n) \) for any given element in \( D^{-m}(R^+_n) \). In order to study the inhomogeneous Dirichlet problem (see (2.5)) we need to define the trace (i.e., restriction in a suitable sense) on \( R^{n-1}_n \) of the functions of \( D^m(R^+_n) \).

**Definition 2.1.** Let \( s \) be a real number and denote by \( D^s(R^{n-1}) \) the completion of \( C^\infty(R^{n-1}) \) with respect to the norm

\[
\| \phi \|_{D^s(R^{n-1})} = \left( \int_{R^{n-1}} \left| \xi' \right|^{2s} \left| \hat{\phi}(\xi') \right|^2 d\xi' \right)^{1/2}, \quad \phi \in C^\infty_c(R^{n-1}),
\]

where \( \hat{\phi} \) denotes the Fourier transform of \( \phi \) and \( \xi' = (\xi_1, \cdots, \xi_{n-1}) \).

One can see [1] that for \(- (n-1)/2 < s < (n-1)/2\), \( D^s(R^{n-1}) \) is a subspace of \( S'(R^{n-1}) \), the space of tempered distributions in \( R^{n-1} \); also the dual of \( D^s(R^{n-1}) \) is \( D^{-s}(R^{n-1}) \).

**Theorem 2.2.** There is a continuous linear map

\[
\gamma = (\gamma_0, \cdots, \gamma_{m-1}) : D^m(R^+_n) \rightarrow \prod_{j=0}^{m-1} D^{m-j-1/2}(R^{n-1})
\]

with the following properties:

(i) for all functions \( \phi \), infinitely differentiable and with compact support in \( \overline{R^+_n} \), \( \gamma \phi = (\partial^j \phi / \partial x^j)(x', 0) \), where \( x' \) denotes \( (x_1, \cdots, x_{n-1}) \);
(ii) \( \gamma \) is onto;
(iii) \( \gamma^{-1}(0) = D^m_0(R^+_n) \).
Using this theorem and Corollary 1.1 we easily derive the following result.

**Theorem 2.3.** \((A, \gamma)\) is an isomorphism of \(D^m(R^n_+)\) onto \(D^{-m}(R^n_+) \times \prod_{j=0}^{m-1} D^{-j-1/2}(R^{n-1})\).

In other words, the inhomogeneous Dirichlet problem

\[
Au = f, \quad \gamma_i u = g_i, \quad 0 \leq j \leq m - 1,
\]

has a unique solution in \(D^m(R^n_+)\) for any given \(f \in D^{-m}(R^n_+)\) and \(g_i \in \prod_{j=0}^{m-1} D^{-j-1/2}(R^{n-1})\).

3. **Regularization.** For any open subset \(\Omega\) of \(R^n\) let \(W^{k,q}(\Omega)\) be the Sobolev space of functions \(u \in L^q(\Omega)\) with derivatives \(D^k u\), in the sense of distributions, belonging to \(L^q(\Omega)\) for \(|\phi| \leq k\). When \(q = 2\), we denote \(W^{k,2}(\Omega)\) by \(H^k(\Omega)\).

**Definition 3.1.** We denote by \(D^{k,m}(R^n_+)\) \((k\ \text{integer} \geq 0)\) the space

\[D^{k,m}(R^n_+) = \{ u \in W^{k,q}(R^n_+) : D^p u \in W^{k,q}(R^{n-1}) \}, 0 \leq |\phi| = j \leq m \},
\]
equipped with its natural norm.

**Theorem 3.3.** If \(f \in D^{-(m-1)}(R^n_+) \cap D^{-m}(R^n_+)\), then the unique solution \(u \in D^m(R^n_+)\) of (2.3) belongs to \(D^{1,m}(R^n_+)\).

4. **Transposition.** Let \(a^*(u, v) = (a(v, u))^{-}\) and let \(A^*\) be the formal adjoint of \(A\). Suppose that \(a^*(u, v)\) verifies condition (2.2), thus all the previous results apply to \(A^*\). In particular, Corollary 2.1 together with Theorem 3.3 hold.

**Theorem 4.1.** \(A^*\) is an isomorphism of \(D^{m,m}(R^n_+) \cap D_0^m(R^n_+)\) onto \(\bigcap_{p=0}^{m} D^{-p}(R^n_+)\). (Here \(D^0(R^n_+) = L^2(R^n_+)\).)

If we transpose this result and use the same trick as in [4] we get

**Theorem 4.2.** \((A, \gamma)\) is an isomorphism of \(H\) onto

\[D^{-m}(R^n_+) \times \prod_{j=0}^{m-1} H^{-j-1/2}(R^{n-1})\],

where \(H\) is the space of functions \(w\) belonging to the dual of \(\bigcap_{p=0}^{m} D^{-p}(R^n_+)\) such that \(Aw \in D^{-m}(R^n_+)\), equipped with its natural norm.

5. **Interpolation.** We can now obtain other isomorphism theorems by interpolating the results of Theorem 2.3 and of Theorem 4.2. If
$A$ and $B$ are two Banach spaces contained in a topological vector space $S$, if $1 < p, q < +\infty$, $\alpha$ and $\beta$ two real numbers such that $1/p + \alpha$ and $1/q + \beta$ lie in the interval $(0, 1)$, we shall consider the trace space $T(p, \alpha; A; q, \beta; B)$, as introduced by Lions in [3]. This space verifies the interpolation property with respect to continuous linear maps [3, Theorem 3.1]. Applying this fact to our situation we get, taking $p = q = 2$, $\alpha = \beta$, after an obvious change of notation:

**Theorem 5.1.** $(A, \gamma)$ is an isomorphism of $T(2, \alpha; D^m(R^n_+), H)$ onto $D^{-m}(R^n_+) \times \prod_{j=0}^{\infty} T(2, \alpha; D^{m-j-\frac{1}{2}}(R^{n-1}), H^{-j+1/2})(R^{n-1}))$.

The significance of this result lies in the fact that we can characterize in an explicit way the elements of $T(2, \alpha; D^{m-j-\frac{1}{2}}, H^{-j+1/2})$.

**Theorem 5.2.** The space $T(2, \alpha; D^{m-j-\frac{1}{2}}(R^n_+), H^{-j+1/2})(R^{n-1}))$ can be identified in the algebraic and topological senses to the completion of $C^\infty(R^n)$ with respect to the norm

$$
\int_{R^{n-1}} |\xi'|^{2(m-j-\frac{1}{2})(1-\theta)}(1 + |\xi'|)^{-2j+1/2}\theta |\hat{\phi}(\xi')|^2 d\xi',
$$

where $\hat{\phi}$ denotes the Fourier transform of $\phi$ and $\theta = 1/2 + \alpha \in (0, 1)$.

**Bibliography**


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