A RADON-NIKODYM THEOREM IN $W^*$-ALGEBRAS

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1. Introduction. The purpose of this paper is to show a Radon-Nikodym theorem in general $W^*$-algebras as follows: Let $M$ be a $W^*$-algebra, and $\phi$, $\psi$ two normal positive linear functionals on $M$ such that $\psi \leq \phi$; then there is a positive element $t_0$ of $M$ with $0 \leq t_0 \leq 1$ satisfying $\psi(x) = \phi(x t_0)$ for all $x \in M$ (Theorem 2). This theorem is the affirmative solution to a problem raised by Dixmier [1, p. 63] and the author [3, p. 1.46 and Question 2 in the appendix]. A less cogent Radon-Nikodym theorem in general $W^*$-algebras has been proved by the author [3, p. 1.46].

2. Theorems. To prove the above theorem, we shall provide some considerations.

Let $M$ be a $W^*$-algebra, $\phi$ a normal positive linear functional on $M$. For $a, x \in M$, put $(Ra\phi)(x) = \phi(xa)$; then $Ra\phi$ is a $\sigma$-continuous linear functional on $M$. Then we shall show

**Proposition 1.** Suppose that $Ra\phi$ is self-adjoint; then we have

$$| (Ra\phi)(h) | \leq \|a\|\phi(h) \text{ for } h \in M.$$  

**Proof.** By the assumption, $(Ra\phi)^*(x) = [(Ra\phi)(x^*)]^- = [\phi(x^*a)]^-$

$$= [\phi((a^*x)^*)]^-=\phi(a^*x) = (Ra\phi)(x) = \phi(xa) \quad \text{for } x \in M.$$  

Hence $\phi(a^*x) = \phi(xa)$, so that $\phi(xa^2) = \phi(xaa) = \phi(a^*xa)$; therefore $Ra^2\phi \geq 0$ and so, analogously, we have $\phi(xa^4) = \phi((a^2)^*xa^2)$.

By the analogous discussion, we have

$$\phi(xa^{n+1}) = \phi((a^2)^*x(a^2)^n) \quad \text{for } x \in M.$$  

Then, for $h \geq 0$,

$$| \phi(ha) | = | \phi(h^{1/2}a^{1/2}) | \leq \phi(h^{1/2}\phi(a^2)a^{1/2})$$

$$= \phi(h^{1/2}\phi(ha^2)^{1/2}) \leq \phi(h^{1/2}\phi((a^2)^*ha^2)^{1/2})^{1/2}$$

$$= \phi(h^{1/2}\phi(h^4)^{1/4}) = \phi(H^{1/2} + 1/\phi(h^4)^{1/4})$$

$$= \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$= \phi(h)^{2^{n-1}} (1/\phi(h^4)^{1/2}) \phi(ha^2)^{1/2} = \phi(h)^{1/2} \phi(ha^2)^{1/2}$$

$$\leq \phi(h)^{1/2} \phi(ha^2)^{1/2} < \phi(ha^2)^{1/2} \quad \text{for } x \in M.$$  

$$\phi(h)^{1/2} \phi(ha^2)^{1/2} \rightarrow \|a\|\phi(h) \quad (n \rightarrow \infty).$$

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Hence we have \( |\phi(ha)| \leq \|a\|\phi(h) \).

This completes the proof.

Now we shall show an application of Proposition 1. For \( b \in M \), we consider a linear functional \( Rb\phi \), then

**Theorem 1.** Let \( Rb\phi = Rv |Rb\phi| \) be the polar decomposition of \( Rb\phi \) (cf. [2], [3]); then the absolute value \( |Rb\phi| \) of \( Rb\phi \) is majorized by \( \|b\|\phi \), that is, \( |Rb\phi| \leq \|b\|\phi \).

**Proof.** Since \( |Rb\phi| = Rv^*(Rb\phi) \) (cf. [2], [3]), \( |Rb\phi| (x) = \phi(xv^*b) \), so that by Proposition 1 we have

\[
|\phi(hv^*b)| = \phi(hv^*b) \leq \|v^*b\|\phi(h) \leq \|b\|\phi(h) \text{ for } h \geq 0 \in M.
\]

This completes the proof.

Now let \( s(\phi) \) be the support of \( \phi \) and we shall consider the \( W^* \)-algebra \( s(\phi)M\phi(\phi) \). Let \( \hat{\phi} \) be the restriction of \( \phi \) on \( s(\phi)M\phi(\phi) \).

Let \( \pi(\hat{\phi}) \) be the \( W^* \)-representation of \( s(\phi)M\phi(\phi) \) on a Hilbert space \( \hat{\phi} \) constructed via \( \phi \), then we can consider \( s(\phi)M\phi(\phi) \) as a concrete \( W^* \)-algebra on the Hilbert space \( \hat{\phi} \). Let \( \xi \) be the image of \( s(\phi) \) in \( \hat{\phi} \), then \( \phi(x) = \langle x\xi, \xi \rangle \) for \( x \in s(\phi)M\phi(\phi) \), where \( \langle , \rangle \) is the inner product of \( \hat{\phi} \).

Let \( \{s(\phi)M\phi(\phi)\}' \) be the commutant of \( s(\phi)M\phi(\phi) \) in \( \hat{\phi} \), then \( \{s(\phi)M\phi(\phi)\}'\xi = \{s(\phi)M\phi(\phi)\)' \xi = \hat{\phi} \), where \( \{ , \} \) is the closed linear subspace of \( \hat{\phi} \) generated by \( \langle , \rangle \), namely, \( \xi \) is a separating and generating vector of \( s(\phi)M\phi(\phi) \).

Now we shall show

**Theorem 2.** Let \( \psi \) be a normal positive linear functional on \( M \) such that \( \psi \leq \hat{\phi} \); then there is a positive element \( t_0 \) of \( M \) with \( 0 \leq t_0 \leq 1 \) satisfying \( \psi(x) = \phi(t_0xt_0) \) for \( x \in M \).

**Proof.** Let \( \hat{\psi} \) be the restriction of \( \psi \) on \( s(\phi)M\phi(\phi) \); then \( \hat{\psi} \leq \hat{\phi} \) and, therefore, there is a positive element \( h_0 \) with \( \|h_0\| \leq 1 \) of \( \{s(\phi)M\phi(\phi)\}' \) such that \( \hat{\psi}(x) = \langle xh_0\xi, h_0\xi \rangle \) for \( x \in s(\phi)M\phi(\phi) \).

Now we shall consider a \( \sigma \)-continuous linear functional \( f' \) on the \( W^* \)-algebra \( \{s(\phi)M\phi(\phi)\}' \) as follows: \( f'(y') = \langle y'h_0\xi, \xi \rangle \) for \( y' \in \{s(\phi)M\phi(\phi)\}' \); then \( f' = Rg' \), where \( g'(y') = \langle y'\xi, \xi \rangle \) for \( y' \in \{s(\phi)M\phi(\phi)\}' \).

Since \( g' \geq 0 \), by Theorem 1, \( |f'| \leq \|h_0\|g' \), so that there is a positive element \( t_0 \) of \( s(\phi)M\phi(\phi) \) with \( 0 \leq t_0 \leq 1 \) such that \( |f'| (y') = \langle y't_0\xi, \xi \rangle \).

Then

\[
|f'| (y') = R_{t_0}(y') = f'(y'v^*) = g'(y'v^*h_0),
\]
where \( R_{f'} \mid f' \mid = f' \) is the polar decomposition of \( f' \).

Hence
\[
\langle y' t_0 \xi, \xi \rangle = \langle y' v'^* h'_0 \xi, \xi \rangle
\]
for \( y' \in \{ s(\phi) M s(\phi) \}' \).

Since \( \{ s(\phi) M s(\phi) \}' \xi = \xi \), we have \( t_0 \xi = v'^* h'_0 \xi \) and so \( v' t_0 \xi = v'^* h'_0 \xi \).

On the other hand,

\[
\langle y' v'^* h'_0 \xi, \xi \rangle = \mid f' \mid \langle y' v' \rangle = R_{f'} \mid f' \mid (y') = f(y') \]

\[
= \langle y' h'_0 \xi, \xi \rangle \quad \text{for} \quad y' \in \{ s(\phi) M s(\phi) \}';
\]
hence \( v'^* h'_0 \xi = h'_0 \xi \) and so \( v' t_0 \xi = h'_0 \xi \). Therefore,

\[
\tilde{\psi}(x) = \langle x h'_0 \xi, h'_0 \xi \rangle
\]
\[
= \langle x v'^* h'_0 \xi, t_0 \xi \rangle = \langle x v'^* v' t_0 \xi, t_0 \xi \rangle
\]
\[
= \langle x v'^* h'_0 \xi, t_0 \xi \rangle = \langle x t_0 \xi, t_0 \xi \rangle
\]
\[
= \langle t_0 x t_0 \xi, \xi \rangle
\]
\[
= \phi(t_0 x t_0) \quad \text{for} \quad x \in s(\phi) M s(\phi).
\]

Now we have

\[
\psi(x) = \psi(s(\phi) x s(\phi)) = \tilde{\psi}(s(\phi) x s(\phi))
\]
\[
= \phi(t_0 s(\phi) x s(\phi) t_0)
\]
\[
= \phi(t_0 x t_0) \quad \text{for} \quad x \in M.
\]

This completes the proof.

**References**


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