EXTENSIONS OF HAAR MEASURE FOR COMPACT CONNECTED ABELIAN GROUPS

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We outline in this paper generalizations of some theorems of Hulanicki on the existence of dense subsets of small cardinality in product measure spaces and in compact groups. We then apply a special case of these results to show the existence of the Kakutani-Oxtoby measure for the case of compact connected Abelian topological groups. A more detailed paper will appear later on.

DEFINITION. Let $\mathfrak{a}, \mathfrak{b}$ be collections of nonvoid sets of a space $X$. Then $\mathfrak{a}$ is a weak base for $\mathfrak{b}$ if and only if given $B \in \mathfrak{b}$ there is an $A \in \mathfrak{a}$ such that $A \subseteq B$.

If $A$ is a set then $|A|$ denotes the cardinal of $A$; $\aleph$ will always denote an infinite cardinal.

The following theorem generalizes Hulanicki [7, Theorem 1].

THEOREM 1. Let $X = \prod_{t \in T} X_t$, where $\{(X_t, \mathcal{A}_t): t \in T\}$ is a family of measurable spaces, each having a weak base of cardinal at most $\aleph^0$, and $|T| \leq 2^n$. Then the product measurable space $(X, \mathfrak{A})$ has a weak base $\mathfrak{a}$ for the $\sigma$-field $\mathfrak{b}$ for which $|\mathfrak{a}| \leq \aleph^n$.

The proof uses the following lemma.

LEMMA 1. Let $T$ be any set such that $|T| = 2^n$; then there exists a family $\mathfrak{A}$ of sequences $\{B_i, i \leq n \}$ of pairwise disjoint subsets of $T$ such that

(i) $|\mathfrak{A}| \leq \aleph^{\aleph^0}$,

(ii) for any distinct sequence $\{t_i\}_{i \leq n}$ in $T$, there exists a sequence $\{B_i, i \leq n \} \subset \mathfrak{A}$ such that $t_i \in B_i$ for each $i$.

This lemma can be proved by noticing that there is a 1-1 correspondence of $T$ with $\{-1, 1\}^n$, and this latter set has at most $\aleph^{\aleph^0}$ closed $G_\delta$ sets.

Let $X$ be a topological space. Let $w(X)$ denote the least cardinal of a basis of open sets for $X$. It is not difficult to show that if $H$ is a compact Abelian group and if $w(H) \leq \aleph$, then $H$ has at most $\aleph^{\aleph^0}$ closed $G_\delta$ sets. Thus, trivially, there is a weak base for the Baire sets of $H$ having cardinal at most $\aleph^{\aleph^0}$.

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COROLLARY 1. Let \( G = \bigoplus_{t \in T} H_t \), where each \( H_t = H \), \( H \) is a compact Abelian group, \( w(H) \leq n \), and \( |T| \leq 2^n \). Then \( w(G) \leq 2^n \) and there is a weak base for the Baire sets of \( G \) having cardinal at most \( n^{n^0} \).

Kakutani [8] has shown that if \( w(H) = n \), then \( |H| = 2^n \) (see also [3, 24.47]). Thus the following holds.

COROLLARY 2. Let \( G = \bigoplus_{t \in T} H_t \), where each \( H_t = H \), \( H \) is a compact Abelian group, \( w(H) \leq n \), and \( |T| \leq 2^n \). Then there is a weak base for the Baire sets of \( G \) having cardinal at most \( 2^n \).

COROLLARY 3. Let \( G \) be as in Corollary 2. Then \( G \) has a dense pseudocompact subgroup \( J \) which necessarily has Haar outer measure one and \( |J| \leq 2^n \).

This follows from a theorem of Comfort and Ross [1] which states that a totally bounded group \( G \) is pseudocompact if and only if each nonempty Baire subset of \( \overline{G} \) meets \( G \), where \( \overline{G} \) is the Weil completion of \( G \). We note in the proof that if \( H \) is a compact group and if \( A \subset H \), then \( A \) has Haar outer measure one if and only if \( A \cap B \neq \emptyset \) for each Baire set \( B \) of positive measure.

THEOREM 2. Let \( G \) be a compact Abelian topological group satisfying \( w(G) = 2^n \) for some infinite cardinal number \( n \). Then

(i) \( G \) has a weak base for its Baire sets of cardinal at most \( n^{n^0} \),

(ii) \( G \) contains a dense pseudocompact subgroup \( J \) such that \( |J| \leq n^{n^0} \); necessarily \( J \) has outer measure one.

This theorem is proved by using Corollary 1 and the following theorem of Vilenkin [11]: Let \( G \) be a compact Abelian group. For some cardinal number \( m \), there is a continuous mapping of \( \{-1, 1\}^m \) onto \( G \); \( m \) can be taken to be \( \max[\aleph_0, r] \), where \( r \) is the rank of the character group of \( G \).

We may observe that Theorem 2 is a generalization of a theorem of Hartman and Hulanicki [2]: If \( G \) is a compact group satisfying \( |G| \leq 2^n \) and if the generalized continuum hypothesis holds, then there is a dense subgroup \( H \subset G \) satisfying \( |H| \leq n \). We note here that we did not use the generalized continuum hypothesis. Finally, part (i) of Theorem 2 appears to contain Theorem 2 of Hulanicki [7].

We next prove a special case of Corollary 2. We note that Corollary 2 is an existence theorem. We will now construct a set that is actually a weak base for the closed \( G_\delta \) sets of \( G \) in Corollary 2.

Let \( G \) be as in Corollary 2. Let \( \mathcal{H} \) be the collection of closed \( G_\delta \) sets of \( H \). As above we note that \( |\mathcal{H}| \leq n^{n^0} \). Let \( \mathcal{A} \) be the collection of sequences of pairwise disjoint sets in \( T \) satisfying (i) and (ii) of Lemma 1.
DEFINITION. An \((\mathfrak{A}, \mathfrak{K})\)-cylinder set in \(G\) is a set of the form
\[ M = \bigcap_{i=1}^{n} \{ \bigcap_{t \in T} \pi_{i(t)}^{-1}(N_{i(t)}) \}, \]
where \( \{ B(i) \}_{i=1}^{n} \subseteq \mathfrak{A} \), and for each \( i \), all \( N_{i(t)} = N_i \) for some \( N_i \in \mathfrak{K} \).

Let \( \mathcal{C}_d \) be the collection of all \((\mathfrak{A}, \mathfrak{K})\)-cylinder sets in \( G \). It is immediate from Lemma 1 that \( | \mathcal{C}_d | \leq 2^{n} \).

**Theorem 3.** Let \( G \) be as in Corollary 2. Then \( \mathcal{C}_d \) is a weak base for the closed \( G_\beta \) sets in \( G \).

The proof of this theorem uses the following lemma and the reflexivity of the property of being a weak base.

**Lemma 2.** Let \( G \) be as in Corollary 2. Then the collection \( \mathcal{G}_d \) of all non-void closed \( G_\beta \) sets in \( G \) of the form
\[ \bigcap_{i=1}^{n} \pi_{i(t)}^{-1}(N_{i(t)}) \]
where \( \{ t(i) \}_{i=1}^{n} \subseteq T \), and \( N_{i(t)} \in \mathfrak{K} \) for each \( i \), is a weak base for the closed \( G_\beta \) sets in \( G \).

Kakutani and Oxtoby [10] proved that Haar measure in a compact metric group may be extended to a much larger \( \sigma \)-field of subsets of the group and still remain invariant under group translation and inversion. To be more precise we introduce the following definition.

**Definition.** The character of a measure space \((X, \mathcal{S}, \mu)\) is the smallest cardinal number \( m \) for which there is a subfamily \( \mathcal{R} \subseteq \mathcal{S} \) such that \( |\mathcal{R}| = m \) and such that for each \( S \in \mathcal{S} \) and each \( \epsilon > 0 \), there exists a set \( R \in \mathcal{R} \) satisfying \( \mu(S \Delta R) < \epsilon \).

It is well known that the character of the Haar measure space of a compact infinite metric group is \( \aleph_0 \). Kakutani and Oxtoby showed that there is an extension of Haar measure with character \( 2^\aleph_0 \).

Kakutani and Kodaira [9] showed that there is an extension of Haar measure on the circle of character \( c \). Hulanicki [7], using Theorem 1 of his paper, showed that the method of Kakutani and Kodaira may be used to get an extension of character \( 2^c \).

**Theorem 4.** Let \( H \) be a compact connected Abelian topological group satisfying \( \omega(H) = n \). Then there exists a translation- and inversion-invariant extension of Haar measure on \( H \) of character \( 2^n \).

We remark that for a compact infinite Abelian group \( G \) it is easy to show that the character of the Haar measure space of \( G \) is equal to \( \omega(G) \). Thus the character of the Haar measure space of \( H \) in the above theorem is \( n \). Our method of proof of Theorem 4 is similar to that of Kakutani and Kodaira. We briefly outline the proof in the following theorem and lemmas.

**Theorem 5.** Let \( G \) be as in Corollary 2. Let \( \beta \in T \) be fixed. Let \( \mathcal{C}_d \subseteq \mathcal{C}_d \) consist of those \((\mathfrak{A}, \mathfrak{K})\)-cylinder sets of the form
\[ n \cap \left\{ \cap_{(n) \in B(n)} \pi_{(n)}^{-1}(N_n) \right\} \] that satisfy \( \beta \in B(i) \) for some \( i \) and \( N_i \) has positive Haar measure in \( H \) for this \( i \). Then \( \emptyset_\beta \) is a weak base for the closed \( G_\beta \) sets in \( G \) having positive Haar measure.

**Remark.** It is clear from the construction of \( \emptyset_\beta \) that \( |\emptyset_\beta| \leq 2^n \) and if \( A \subseteq \emptyset_\beta \) then \( \pi_\beta(A) \) has positive Haar measure in \( H_\beta \) and is a closed \( G_\beta \) there (\( \pi_\beta \) is the projection onto \( H_\beta \)).

**Lemma 3.** Let \( G \) be a compact Abelian topological group. Let \( M \subseteq G \) be a set of positive Haar measure. Then \( M \) contains a maximal independent set of elements of infinite order in \( G \).

This lemma is a consequence of a well-known theorem which states (using additive notation) that if \( M \) has positive Haar measure in \( G \) then \( M - M \) contains the identity in its interior. It follows then that the group \( [M] \) generated by \( M \) has finite index in \( G \) if \( G \) is compact and hence every element of infinite order in \( G \) is dependent on \( M \).

**Lemma 4.** Let \( G \) be a compact connected Abelian topological group satisfying \( w(G) = n \). Then every closed \( G_\beta \) set \( M \subseteq G \) having positive Haar measure contains a maximal linearly independent set \( L \) of elements of infinite order in \( G \) and \( |L| = 2^n \).

This lemma follows from Lemma 3, the fact that all maximal linearly independent sets of elements of infinite order have the same cardinality, and a structure theorem of Hulanicki \([5],[6]\) for compact connected Abelian groups. Lemma 4 allows us to carry out a transfinite induction which leads to:

**Lemma 5.** Let \( G \) be a compact connected Abelian group satisfying \( w(G) = n \geq \aleph_0 \). Let \( \{ M_\alpha : \alpha < \omega_m, m = 2^n \} \) be a well-ordered sequence of closed \( G_\beta \) sets of positive Haar measure in \( G \). Then there exists a well-ordered set \( \{ x_\alpha : \alpha < \omega_m \} \) of independent elements of infinite order such that \( x_\alpha \subseteq M_\alpha \) for each \( \alpha < \omega_m \). (The \( M_\alpha \)'s are not necessarily distinct.)

**Remark.** Lemma 5 is true in a more general situation. The same induction will work because of Lemma 3 if the \( M_\alpha \) are measurable with positive measure, \( m \) is at most equal to the cardinal of a maximal independent set of elements of infinite order, and \( G \) is compact Abelian (with no other restrictions).

**Lemma 6.** Let \( H \) be a compact connected Abelian group satisfying \( w(H) = n \geq \aleph_0 \). Let \( G = \bigcup_{\beta \in T} H_\beta \) where each \( H_\beta = H \) and \( |T| = 2^n \). Fix the coordinate \( \beta \in T \). Then there is a set \( V \subseteq G \) of independent elements of infinite order satisfying

(i) \( V \) has Haar outer measure one,
(ii) \( \pi_\beta|_V \) is one-to-one.
This is proved by using Theorem 5 (i.e., projecting onto $H_\beta$ the elements of $\mathcal{P}_\beta$) and then using Lemma 5.

Letting $V_\alpha$ be the free group generated by $V$, and letting $W$ be the free group generated by $\pi_\beta(V)$, it is easy to see that $\pi_\beta$ induces an algebraic isomorphism $\phi$ of $W$ onto $V_\alpha$. Furthermore, $\phi$ may be extended to an algebraic isomorphism of $H$ into $G$ satisfying $\pi_\beta \phi(x) = x$ for all $x \in H_\beta$, because $H$ and $G$ are divisible. It follows that $\phi(H_\beta)$ is a group of outer measure one in $G$. Thus the remainder of the proof of Theorem 4 is a repetition of the final part of the proof of Kakutani and Kodaira [9] for the circle.

REMARK. One could use the method of proof outlined above without Theorem 5 to show the existence of an extension of Haar measure of character $2^n$.

NOTE. Since this work was completed, Hewitt and Ross [4], have generalized and simplified Theorem 4; their theorem implies Theorem 4 for all compact Abelian groups, and uses our Theorem 2, Lemma 3, and Lemma 5 with the remark following it.

BIBLIOGRAPHY


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