MULTIPLIERS OF FOURIER TRANSFORM
IN A HALF-SPACE

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1. Let \((x, y)\) denote points in \(\mathbb{R}^n\) where \(x = (x_1, \ldots, x_{n-1})\), \(y = x_n\). Points of the dual space are denoted by \((\xi, \eta)\). Let \(Y_+\) be the characteristic function of the half space \(R^+_n = \{(x, y) | y \geq 0\}\). Let \(M(\xi, \eta)\) be an \(m \times m\) matrix-valued function whose entries are homogeneous functions:

\[ M_{ij}(\lambda \xi, \lambda \eta) = M_{ij}(\xi, \eta), \quad \lambda > 0, 1 \leq i, j \leq m. \]

Assume further that \(M(\xi, \eta)\) is continuous and nonsingular for \((\xi, \eta) \neq 0\). Consider the bounded operator \(M\) in the space \((L^2(\mathbb{R}^+_n))^m\) (with the natural norm denoted by \(\| \|\)):

\[ Mu = Y_+ \mathcal{F}^{-1} \{M(\xi, \eta)(\mathcal{F}u)(\xi, \eta)\}, \quad u \in (L^2(\mathbb{R}^+_n))^m, \]

where \(\mathcal{F}\) \((\mathcal{F}^{-1})\) denotes the direct (inverse) Fourier transform with respect to all variables. \(\mathcal{F}_y\) \((\mathcal{F}_x)\) will denote the transform with respect to \(y\) or \(x\) alone. The one-dimensional operator \(M_\xi\) is similarly defined in \((L^2(\mathbb{R}^+_n))^m\) with the multiplier \(M(\xi, \eta), \xi\) fixed:

\[ M_{\xi}v = Y_+ \mathcal{F}_y^{-1} \{M(\xi, \eta)(\mathcal{F}_y v)(\eta)\}. \]

Our main results in this note are the following lemma and theorem.

**Lemma.** The estimate

\[ ||u|| \leq C||Mu||, \quad u \in (L^2(\mathbb{R}^+_n))^m \]

holds if and only if for all \(|\xi| = 1\) (uniformly)

\[ ||v|| \leq C||M_{\xi}v||, \quad v \in (L^2(\mathbb{R}^+_n))^m. \]

For the scalar case \((m = 1)\), we have

**Theorem.** Let \(M(\xi, \eta)\) be a homogeneous function continuous and nonvanishing for \((\xi, \eta) \neq 0\). Let

\[ -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \arg M(\xi, \eta) = k + \theta, \quad k \text{ integer}, -1/2 < \theta \leq 1/2. \]

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If \( \theta \neq 1/2 \), then \( M \) has a closed range and is injective if \( k \geq 0 \), surjective if \( k \leq 0 \).

**Remarks.**

1. The a priori \( L^2 \)-estimates for mixed elliptic problems can be reduced to the validity of (3) \([2], [3]\).

2. For \( n = 1 \), \( M = M(\eta) \) is determined by \( M(1) \) and \( M(-1) \). The operator \( M \) is then a singular integral operator (with Cauchy kernel) on a half-line. For this case it was shown \([4]\), cf. also \([1], [2], [5]\) that \( M \) is invertible if and only if the matrix \( M(1)^{-1}M(-1) \) does not have real negative eigenvalues.

2. **Proof of the Lemma.** Assume first that (4) holds, and apply it to \( M \left( \frac{\xi}{|\xi|}, \eta \right) \) and \( v(y) = \langle \mathfrak{F}u \rangle \left( \frac{\xi}{|\xi|}, \frac{y}{|\xi|} \right) \).

We get
\[
\int_{0}^{\infty} \left| \mathfrak{F}u \left( \frac{\xi}{|\xi|}, \frac{y}{|\xi|} \right) \right|^2 dy \leq C^2 \int_{0}^{\infty} \left| \mathfrak{F}^{-1}M \left( \frac{\xi}{|\xi|}, \eta \right) \mathfrak{F}v \left[ \mathfrak{F}u \left( \frac{\xi}{|\xi|}, \frac{y}{|\xi|} \right) \right] \right|^2 dy
\]
\[
= C^2 \int_{0}^{\infty} \left| \mathfrak{F}^{-1}M \mathfrak{F}u \left( \frac{\xi}{|\xi|}, \frac{y}{|\xi|} \right) \right|^2 dy
\]
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\]

After changing variables on both sides (put \( \xi = y/|\xi| \)), cancelling \( |\xi| \) and integrating with respect to \( \xi \), we have
\[
\int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} \left| \mathfrak{F}u(\xi, y) \right|^2 dy d\xi \leq C^2 \int_{\mathbb{R}^{n-1}} \int_{0}^{\infty} \left| \mathfrak{F}^{-1}M \mathfrak{F}u(\xi, y) \right|^2 dy d\xi.
\]

Using Parseval's identity (for \( \mathfrak{F}^{-1} \)) we obtain (3).

Assume now that (4) does not hold for some \( \xi \). Then for any \( \epsilon > 0 \) there is a \( v_\epsilon(y) \in L^2(K_\epsilon) \) such that \( \|v_\epsilon(y)\| = 1 \) and \( \|M_{\xi}v_\epsilon(y)\| \leq \epsilon/2 \). It is easily seen that \( \|M_{\xi}v_\epsilon\| \leq \epsilon \) if \( |\xi - \xi_\epsilon| < \delta \) and \( \delta = \delta(\epsilon) \) is sufficiently small. Let now \( w(\xi) \) be the characteristic function of the unit cube and
\[
u_\epsilon(x, y) = v_\epsilon(y)(2\delta)^{-(n-1)/2}\mathfrak{F}^{-1}w \left( \frac{\xi - \xi}{\delta} \right).
\]
Then $||u_c|| = 1$ and $||Mu_c|| \leq \varepsilon$, contradicting (3).

**Proof of the Theorem.** Solving $M\varphi = w$ is readily seen to be equivalent (via Fourier transform) to solving the Riemann-Hilbert problem

$$
\Phi^-(\eta) = M(\xi, \eta)\Phi^+(\eta) + \Psi(\eta)
$$

where $\Phi^\pm$ are sought in $(H^2_\pm(R^1))^m$, the space of transforms of $L^2$-vector functions supported in $R^1_{\pm}$, and $\Psi(\eta)$ is a given $L^2$-function. In the scalar case ($m = 1$) this was done by Widom [5, Theorem 3.2]. It follows from Widom's results that if in (5) $\theta \neq 1/2$ and $k \geq 0$ then $M_\xi$ is injective and has a closed range for every $\xi \neq 0$, so that (4) is satisfied. It is clear that (4) is satisfied uniformly on the compact set $|\xi| = 1$, and by the Lemma we obtain that $M$ is injective and has a closed range. If $\theta \neq 1/2$ and $k \leq 0$ in (5), a consideration of $M^*$, the adjoint of $M$, which corresponds to the multiplier $M(\xi, \eta)$, shows that $M$ is surjective.

**Remark.** It is easily seen that the expression (5) does not depend on $\xi$. Indeed, the homogeneity of $M$ implies that $\lim_{s \to \pm\infty} M(\xi, \eta) = M(0, \pm 1)$, for any $\xi \neq 0$. This means that $\theta$ in (5) is the same for all $\xi$, and by continuity, the same is true for the integer $k$.

**References**


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