VARIATIONAL METHODS FOR NONLINEAR ELLIPTIC EIGENVALUE PROBLEMS

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In the present note, we give a simple general proof for the existence of solutions of the following two types of variational problems:

**PROBLEM A.** To minimize $\int_\Omega F(x, u, \cdots, D^m u) dx$ over a subspace $V$ of $W^{m,r}(\Omega)$.

**PROBLEM B.** To minimize $\int_\Omega F(x, u, \cdots, D^m u) dx$ for $u$ in $V$ with $\int_\Omega G(x, u, \cdots, D^{m-1} u) dx = c$.

The solution of the first problem yields a weak solution of a corresponding elliptic boundary-value problem for the Euler-Lagrange equation

$$ (1) \quad Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^m F_{\alpha}(x, u, \cdots, D^m u) = 0. $$

From the solution of the second problem, we obtain a solution under corresponding boundary conditions of the nonlinear eigenvalue problem.

$$ (2) \quad Au = \lambda \left\{ \sum_{|\beta| \leq m-1} (-1)^{|\beta|} G_{\beta}(x, u, \cdots, D^{m-1} u) \right\} = \lambda Bu, \quad \lambda \in \mathbb{R}^1. $$

In §1, we give a complete self-contained treatment of the existence of minima of functionals on reflexive Banach spaces, a treatment which extends and strengthens earlier studies by Lusternik, E. Rothe, Vainberg, and others (see [6], [11], [12], [14], [15]). In §2, we apply the results of §1 to Problems A and B, above. In the case of Problem A, we strengthen and simplify results of Morrey [10] and Smale [13]. The relation of the resulting existence theorem for the solution of the variational boundary-value problem for equation (1) to those obtained by the writer in [2], [3], [4] by operator methods (as well as unpublished results of Leray and Lions) and the results of Višik [16] using other analytical methods, is discussed in detail in [5]. Special cases of the eigenvalue problem treated in Problem B have been treated for $A$ linear by Levinson [7] with $A = \Delta$ on $\mathbb{R}^2$, and by Berger [1] for general linear $A$.

1. Abstract variational problems. Let $V$ be a real Banach space. Strong convergence in $V$ is denoted by $\to$, weak convergence by $\rightharpoonup$. We consider two functions
\[ \Phi : V \times V \to \mathbb{R}, \]
\[ g : V \to \mathbb{R}, \]

and define \( f : V \to \mathbb{R} \) by \( f(v) = \Phi(v, v), \quad v \in V \).

The function \( \Phi \) is said to be \emph{semi-convex} if both of the following conditions hold:

(a) \( \text{For each } v \text{ in } V \text{ and } c \in \mathbb{R}^1, \text{ the subset } W_{e,v} \text{ of } V \text{ given by} \)

\[ W_{e,v} = \{ u \mid u \in V, \Phi(u, v) \leq c \} \]

is convex.

(b) \( \text{For each bounded set } B \text{ of } V \text{ and each sequence } \{v_j\} \text{ in } V \text{ with} \)
\[ v_j \to v, \quad \Phi(u, v_j) \to \Phi(u, v) \text{ uniformly for } u \text{ in } B. \]

For fixed \( v \text{ in } V \), \( \Phi(\cdot, v) \) is continuous in the strong topology of \( V \).

**Theorem 1.** Let \( V \) be a reflexive Banach space, \( \Phi \) a semi-convex real function on \( V \times V \). Let \( f(v) = \Phi(v, v) \) for \( v \) in \( V \), and let \( C \) be a weakly closed bounded subset of \( V \). Then \( f \) is bounded from below on \( C \) and assumes its minimum on \( C \).

**Proof of Theorem 1.** We may choose a sequence \( \{u_j\} \) from the bounded weakly closed set \( C \) such that

\[ f(u_j) = \Phi(u_j, u_j) \to c_0 = \text{g.l.b.}_u f(u), \]

while

\[ u_j \to u_0, \quad u_0 \in C. \]

By property (b) of semi-convexity and the boundedness of \( C \),

\[ \Phi(u_j, u_j) - \Phi(u_0, u_j) \to 0. \]

Hence

\[ \Phi(u_0, u_j) \to c_0. \]

Let \( c \) be any real number with \( c > c_0 \). Then for \( j \geq j_0, \ u_j \in W_{e,u_0} \) as defined by equation (3) above. \( W_{e,u_0} \) is convex by property (a), and is closed by the second part of property (b) of semi-convexity. Hence \( W_{e,u_0} \) is weakly closed. Since \( u_j \to u_0, \ u_0 \in W_{e,u_0} \), i.e., \( \Phi(u_0, u_0) \leq c \).

Since \( c \) was any number \( > c_0 \), it follows that \( c_0 > -\infty \) and \( f(u_0) = c_0 \).

Q.E.D.

As corollaries of Theorem 1, we have the following:

**Theorem 2.** Let \( \Phi \) be a semi-convex real function on \( V \times V \), where \( V \) is a reflexive \( B \)-space, and for \( v \) in \( V \), let \( f(v) = \Phi(v, v) \). If \( f(v) \to +\infty \) as \( \|v\| \to +\infty \), then \( f \) assumes a minimum on \( V \).
PROOF OF THEOREM 2. Set \( C = \{ v : \| v \| \leq R \} \) for \( R \) sufficiently large.

THEOREM 3. Let \( V \) be a reflexive B-space, \( \Phi \) a real semi-convex function on \( V \times V \), and \( f(v) = \Phi(v, v) \) for \( v \) in \( V \). Let \( g \) be a weakly continuous real function on \( V \). Let \( C = \{ u : g(u) = c \} \) for a fixed \( c \) in \( R^1 \) and suppose that \( f(u) \to +\infty \) as \( \| u \| \to +\infty \) on \( C \). Then \( f \) assumes a minimum on \( C \).

PROOF OF THEOREM 3. \( C \cap \{ u : \| u \| \leq R \} \) is weakly closed and bounded for all \( R > 0 \).

Let \( V^* \) be the adjoint space of \( V \), \((w, u)\) the pairing between \( w \) in \( V^* \) and \( u \) in \( V \).

If \( g : V \to R^1 \), \( g \) is said to be differentiable at \( v_0 \) in \( V \) if there exists an element \( g'(v_0) \) in \( V^* \) such that for all \( h \) in \( V \)

\[
g(v_0 + h) = g(v_0) + (g'(v_0), h) + \varepsilon(h)
\]

where \( \varepsilon(h) = o(\| h \|) \) as \( \| h \| \to 0 \). If \( \Phi : V \times V \to R^1 \), \( \Phi \) is differentiable at \((v_1, v_2)\) if there exists a pair \( w_1, w_2 \) in \( V^* \) such that

\[
\Phi(v_1 + h_1, v_2 + h_2) = \Phi(v_1, v_2) + (w_1, h_1) + (w_2, h_2) + o(\| h_1 \| + \| h_2 \|),
\]

and we set \( w_1 = \Phi'_1(v_1, v_2), w_2 = \Phi'_2(v_1, v_2) \). If \( \Phi \) is differentiable at \((v_0, v_0)\) and \( f(v) = \Phi(v, v) \), then \( f \) is differentiable at \( v_0 \) and

\[
f'(v_0) = \Phi'_1(v_0, v_0) + \Phi'_2(v_0, v_0).
\]

THEOREM 4. Let \( V \) be a Banach space, \( f \) and \( g \) two real functions on \( V \) with \( f \) and \( g \) differentiable at \( v_0 \), \( g'(v_0) \neq 0 \). If \( f \) has a local minimum at \( v_0 \) with respect to the set \( C = \{ v : g(v) = g(v_0) \} \), then there exists \( \lambda \) in \( R^1 \) such that \( f'(v_0) = \lambda g'(v_0) \).

PROOF OF THEOREM 4. Let \( V_1 = \{ v : v \in V, \ (g'(v_0), v) = 0 \} \), and choose \( u_0 \) in \( V \) such that \( (g'(v_0), u_0) = 1 \). If \( v \) is any element of \( V_1 \) with \( \| v \| = 1 \) and \( \varepsilon \) and \( r \) are real numbers with \( |\varepsilon|, |r| \) sufficiently small, then

\[
f(v_0) \leq f(v_0 + \varepsilon v + ru_0)
\]
promised that \( g(v_0) = g(v_0 + \varepsilon v + ru_0) \). We know that

\[
g(v_0 + \varepsilon v + ru_0) = g(v_0) + \varepsilon(g'(v_0), v) + r(g'(v_0), u_0) + s(\varepsilon, r, v)
\]

where for each fixed \( v \) in \( V_1 \), \( s(\varepsilon, r, v) = o(|\varepsilon| + |r|) \). Consider \( r \) on the interval \( [-\frac{1}{2}|\varepsilon|, +\frac{1}{2}|\varepsilon|] \), and the quantity \( r + s(\varepsilon, r, v) \) with \( \varepsilon \neq 0 \) and \( v \) fixed. For \( |\varepsilon| \) sufficiently small, \( r + s(\varepsilon, r, v) \) is negative at the left endpoint, positive at the right, and continuous in \( r \). We may
choose a value of \( r(\varepsilon, v) \) in the interval to make \( r + s(\varepsilon, r, v) = 0 \), and hence \( |r(\varepsilon, v)| = o(\varepsilon) \). For this choice of \( r \), we have

\[
f(v_0) \leq f(v_0 + \varepsilon v + ru_0) = f(v_0) + \varepsilon(f'(v_0), v) + r(f'(v_0), u_0) + o(\varepsilon + |r|)
\]

so that

\[
e(f'(v_0), v) \geq -o(|\varepsilon|), \quad |\varepsilon| \to 0.
\]

Hence \( (f'(v_0), v) = 0 \) for all \( v \) in \( V_1 \) and \( f'(v_0) = \lambda g'(v_0) \), for some \( \lambda \) in \( \mathbb{R}^1 \).

**Remark.** In [6] and [14], Theorem 4 is called Lusternik's principle and proofs are given for special cases.

**Theorem 5.** Let \( V \) be a reflexive Banach space, \( \Phi \) a semi-convex real function on \( V \times V \), \( g \) a weakly continuous real function on \( V \), \( f(v) = \Phi(v, v) \) for \( v \) in \( V \). Suppose that \( f \) and \( g \) are differentiable on \( V \), that for a given constant \( c \) in \( \mathbb{R}^1 \) the set \( C = \{ v \mid g(v) = c \} \) is nonempty, and that \( g'(v) \neq 0 \) for \( v \) in \( C \). Suppose further that \( f(v) \to +\infty \) as \( \|v\| \to +\infty \) on \( C \), then there exists \( v_0 \) in \( C \) and \( \lambda \) in \( \mathbb{R}^1 \) such that \( f'(v_0) = \lambda g'(v_0) \).

Theorem 5 is an immediate consequence of Theorems 3 and 4.

A useful complement to Theorem 5 is the following:

**Theorem 6.** A sufficient condition for condition (a) for semi-convexity to hold is that \( \Phi \) be everywhere differentiable on \( V \times V \) and that \( \Phi'_1(u, v) \) be monotone in \( u \) for fixed \( v \), i.e., for all \( u_0, u_1 \) in \( V \),

\[
(\Phi'_1(u_1, v) - \Phi'_1(u_0, v), u_1 - u_0) \geq 0.
\]

**Proof of Theorem 6.** Let \( u_0, u_1, \) and \( v \) be elements of \( V \), \( 0 \leq \lambda \leq 1 \). Let \( u_\lambda = \lambda u_1 + (1 - \lambda)u_0 \). To prove condition (a), it suffices to show \( \Phi(u, v) \) convex in \( u \) for fixed \( v \). Set

\[
h(\lambda) = \Phi(u_\lambda, v) = \lambda \Phi(u_1, v) - (1 - \lambda)\Phi(u_0, v).
\]

It suffices to show that \( h(\lambda) \leq 0 \) for \( 0 \leq \lambda \leq 1 \). Since \( h(0) = h(1) = 0 \), it suffices to show that \( h'(\lambda) \) is nondecreasing on the interval. However,

\[
h'(\lambda) = (\Phi'_1(u_\lambda, v), u_1 - u_0) - \Phi(u_1, v) + \Phi(u_0, v),
\]

so that for \( \lambda < \xi \),

\[
h'(\xi) - h'(\lambda) = (\Phi'_1(u_\xi, v) - \Phi'_1(u_\lambda, v), u_1 - u_0)
\]

\[
= (\xi - \lambda)^{-1}(\Phi'_1(u_\xi, v) - \Phi'_1(u_\lambda, v), u_\xi - u_\lambda) \geq 0. \quad \text{Q.E.D.}
\]

**Remark.** Connections between monotonicity of the gradient and convexity of the functional have been remarked in Minty [9] and
implicitlly in Vainberg and Kachurovski [15]. Monotone operators between \( V \) and \( V^* \) have been studied in Browder [2], [4] and Minty [8].

2. **Nonlinear eigenvalue problems.** We adopt the notation of [2] and [3] in general, except that all our functions will be real—rather than complex-valued. Let \( \Omega \) be a bounded, smoothly bounded open set in \( \mathbb{R}^n \), \( n \geq 1 \), \( D^a \) the elementary differential operator \( (\partial/\partial x_1)^{a_1} \cdots (\partial/\partial x_n)^{a_n} \). We assume that we are given positive integers \( r \) and \( m \), a real number \( p \) with \( 1 < p < \infty \), and a closed subspace \( V \) of the reflexive Banach space \( W^{m,p}(\Omega) \) of \( r \)-vector functions \( u \) on \( \Omega \) such that \( D^a u \in L^p(\Omega) \) for all \( \alpha \) with \( |\alpha| \leq m \). Let \( \langle , \rangle \) denote the natural inner product in \( \mathbb{R}^r \) and for two real-valued \( r \)-vector functions \( u \) and \( v \) on \( \Omega \), set

\[
[u, v] = \int_\Omega \langle u(x), v(x) \rangle \, dx,
\]

where the integration is taken with respect to Lebesgue \( n \)-measure.

Let \( \xi = \{\xi_\alpha \mid |\alpha| = m\} \) and \( \psi = \{\psi_\xi \mid |\xi| \leq m - 1\} \) be elements of the real vector spaces \( \mathbb{R}^N \) and \( \mathbb{R}^M \), respectively, where for each \( \alpha \) and \( \xi \), \( \xi_\alpha \) and \( \psi_\xi \) are real \( r \)-vectors. We assume that we are given two functions

\[
F(x, \psi, \xi), \quad G(x, \psi)
\]

defined on \( \Omega \times \mathbb{R}^M \times \mathbb{R}^N \) and \( \Omega \times \mathbb{R}^M \), respectively, measurable in \( x \) and \( C^1 \) in \( (\psi, \xi) \) or \( \psi \). We let \( F_\xi \), \( F_\psi \), and \( G_\xi \) denote the appropriate partial gradients of the functions \( F \) and \( G \) with respect to \( \xi_\alpha \) and \( \psi_\xi \).

We suppose that \( F \) and \( G \) satisfy the following system of inequalities:

\[
| F(x, \psi, \xi) | \leq c(\eta) \left\{ g(x) + |\xi|^p + \sum_{|\xi| \leq m-1} |\psi_\xi|^p_\xi \right\},
\]

\[
| G(x, \psi) | \leq c(\eta) \left\{ g(x) + \sum_{|\xi| \leq m-1} |\psi_\xi|^p_\xi \right\},
\]

where \( g \in L^p(\Omega) \), \( p_\xi \) are exponents satisfying the inequalities

\[
(n - p(m - |\xi|))p_\xi < np \quad \text{if} \quad n - p(m - |\xi|) > 0
\]

and \( c(\eta) \) is a continuous function of \( \eta = \{\psi_\xi \mid |\xi| < n/p - m\} \).

For each \( u \) in \( V \), let

\[
\xi(u) = \{D^\alpha u \mid |\alpha| = m\}, \quad \psi(u) = \{D^\xi u \mid |\xi| \leq m - 1\}.
\]

Then the functionals
\[ \Phi(u, v) = \int_{\Omega} F(x, \psi(v), \xi(u)) \, dx \]

and

\[ g(u) = \int_{\Omega} G(x, \psi(u)) \, dx \]

are well defined and continuous on \( V \times V \) and \( V \), respectively, \( g \) is weakly continuous on \( V \), and \( \Phi \) satisfies condition (b) for semi-convexity. If we assume further that:

\[ |F_\alpha(x, \psi, \xi)| \leq c(\eta) \left\{ g_1(x) + |\xi|^{p-1} + \sum_{|\ell| \leq m-1} |\psi_\ell|^{q_\ell} \right\}, \]

\[ |F_\beta(x, \psi, \xi)| \leq c(\eta) \left\{ g_1(x) + |\xi|^{q_\beta} + \sum_{|\ell| \leq m-1} |\psi_\ell|^{q_\ell} \right\}, \]

\[ |G(x, \psi)| \leq c(\eta) \left\{ g_1(x) + \sum_{|\ell| \leq m-1} |\psi_\ell|^{q_\ell} \right\}, \]

where \( g_1 \in L^{p'-1}(\Omega) \) and \( p_\ell', q_\beta \), and \( q_\ell \) are exponents satisfying the inequalities

\[ (n - p(m - |\xi|))p_\ell' \leq n(p - 1), \quad \text{if} \quad n - p(m - |\xi|) > 0, \]

\[ (n - p(m - |\xi|))q_\beta \leq n(p - 1) + p(m - |\beta|), \]

\[ nq_\beta \leq n(p - 1) + p(m - |\beta|) \]

then the functionals \( \Phi \) and \( g \) are everywhere once differentiable with

\[ (\Phi'_1(v_1, v_2), u) = \sum_{|\alpha| = m} [F_\alpha(x, \psi(v_2), \xi(v_1)), D^\alpha u], \]

\[ (\Phi'_2(v_1, v_2), u) = \sum_{|\ell| \leq m-1} [F_\ell(x, \psi(v_2), \xi(v_1)), D^\ell u], \]

\[ (g'(v), u) = \sum_{|\ell| \leq m-1} [G_\ell(x, \psi(v)), D^\ell u]. \]

If we assume that each \( F_\alpha \) is itself differentiable in \( \xi \) and that the following semi-ellipticity condition holds:

\[ \sum_{|\alpha|, |\beta| = m} \langle F_\alpha(x, \psi, \xi) \eta_\alpha, \eta_\beta \rangle \geq 0, \]

for all \( \psi \) in \( R^M \), \( x \) in \( \Omega \), and \( \xi \) and \( \eta \) in \( R^N \) (where \( F_\alpha \) is the gradient of \( F_\alpha \) with respect to \( \xi_\beta \), which we assume to exist); then \( \Phi \) will satisfy the monotonicity condition of Theorem 6 and thereby condition (a) for semi-convexity.

Applying Theorem 2, we have:
Theorem 7. If \( F \) satisfies the inequalities imposed on it in (I), (II), and (III) and if \( f(v) = \Phi(v, v) \to +\infty \) as \( \|v\| \to +\infty \), then \( f \) has a minimum on \( V \) which is a variational solution in the sense of [2] of the Euler-Lagrange equation
\[
Au = \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta F_\phi(x, \psi(u), \xi(u)) = 0.
\]

Applying Theorem 5, we have:

Theorem 8. If \( F \) and \( G \) satisfy (I), (II), and (III), if the set \( C = \{ v | v \in V, g(v) = c \} \) is nonempty and \( g'(v) \neq 0 \) on \( C \), and if, finally, \( f(v) \to +\infty \) as \( \|v\| \to +\infty \) on \( C \), then \( f \) has a minimum on \( C \) which is a variational solution of the appropriate boundary-value problem for the eigenvalue problem
\[
Au = \lambda Bu = \lambda \left\{ \sum_{|\beta| \leq m-1} (-1)^{|\beta|} D^\beta G_\xi(x, \psi(u)) \right\}.
\]

We complete our considerations with the following:

Theorem 9. (a) If \( V \) is \( W_0^{m,p}(\Omega) \), the closure of \( C_0^\infty(\Omega) \) in \( W^{m,p}(\Omega) \), the boundary conditions in Theorems 7 and 8 are those of the homogeneous Dirichlet problem.

(b) Condition (III) can be replaced by the weaker integral condition
\[
\sum_{|\beta| \leq m} [F_{\phi_\beta}(x, \psi(v)), \xi(v)] D^\beta u, D^\beta u \leq -c\|u\|_m^p.
\]

(c) Theorem 8 can be specialized to hold under the following more intuitive restrictions than (I) and (II): namely, \( |F| \leq c \{ 1 + |\xi|^p + |\psi|^p \} \), \( |F_{\phi}| + |F_\phi| \leq c \{ 1 + |\xi|^{p-1} + |\psi|^{p-1} \} \), \( G = u^q \) with \( q < np(n-pm)^{-1} \) for \( n > pm \) and \( G \) an arbitrary continuous function of \( u \) for \( n < pm \).

The regularity of the solutions of Theorems 7 and 8 can be derived from known results for linear and mildly nonlinear equations \( A \) as well as for the case \( m=1, r=1 \).

Bibliography


