A GENERALIZATION OF THE HILTON-MILNOR THEOREM

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The Hilton-Milnor theorem states that \( \Omega \bigvee_{i=1}^{\infty} \Sigma X_i \) is homotopy equivalent to a weak infinite product, \( \prod_{i=1}^{\infty} \Omega \Sigma X_i \), where each \( X_i, i > n \), is a smash product of the \( X_i \)'s, \( i \leq n \). In this note we extend this theorem to the 'wedges' lying between \( \bigvee_{i=1}^{\infty} \Sigma X_i \) and \( \prod_{i=1}^{n} \Sigma X_i \).

It will be assumed that all spaces are connected countable CW-complexes with base points. \( T_1(X_1, \ldots, X_n) \) is the subset of \( X_1 \times \cdots \times X_n \) consisting of those points with at least \( i \) coordinates at base points. \( T_0 \) is the cartesian product and \( T_{n-1} \) is the space studied by Hilton and Milnor. \( T_{n-1} \) will also be denoted by \( \bigvee_{i=1}^{n} X_i \).

The smash product \( \Lambda(X_1, \ldots, X_n) \) is the quotient space \( T_0(X_1, \ldots, X_n)/T_1(X_1, \ldots, X_n) \). Define \( X^{(n)} \) inductively by \( X^{(0)} = S^0 \) and \( X^{(n)} = \Lambda(X^{(n-1)}, X) \), for \( n > 0 \).

The \( n \)-fold suspension, \( \Sigma^n X \), is defined to be \( \Lambda(S^n, X) \). The loop space of \( X \), \( \Omega X \), is the set of maps, \( f: I \rightarrow X \), such that \( f(0) = f(1) = * \). We shall abbreviate \( (\Sigma X_1, \ldots, \Sigma X_n) \) and \( (\Omega X_1, \ldots, \Omega X_n) \) by \( \Sigma(X_1, \ldots, X_n) \) and \( \Omega(X_1, \ldots, X_n) \), respectively.

\textbf{Theorem 1.} \( \Omega T \Sigma(X_1, \ldots, X_n) \) is homotopy equivalent to a weak infinite product, \( \prod_{i=1}^{\infty} \Omega \Sigma X_i \), where each \( X_i \) is equal to \( \Sigma^r \Lambda(X_i^{(r)}, \ldots, X_i^{(s)}) \) for some \( (n+1) \)-tuple, \( (r, j_1, \ldots, j_n) \), depending upon \( j \). Moreover, the set of \( (n+1) \)-tuples over which the product is taken is computable.

If \( i = n - 1 \), Theorem 1 is the Hilton-Milnor theorem. It was proven in [1] by Hilton when the \( X_i \) are spheres and extended to the general case by Milnor [2].

We shall sketch the proof of Theorem 1, when \( n - i \geq 2 \). The details will appear in [3].

The inclusion map \( j: T_i(X_1, \ldots, X_n) \rightarrow T_0(X_1, \ldots, X_n) \) may be replaced by a homotopy equivalent fibre map, \( p: E \rightarrow T_0 \), with fibre \( F_i \). It is easily seen that when \( n - i \geq 2 \), the short exact sequence

\[ * \rightarrow \Omega F_i \rightarrow \Omega E \rightarrow \Omega T_0 \rightarrow * \]

splits yielding:

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LEMMA 1. \( \Omega T_i(X_1, \cdots, X_n) \sim \Omega X_1 \times \cdots \times \Omega X_n \times \Omega F_i. \)

Thus an analysis of \( \Omega T_i \) depends upon a study of \( F_i \). Standard homotopy methods are applied and it is shown that

THEOREM 2. \( F_i \) is homotopy equivalent to

\[
\bigvee_s (\Sigma^{s-1} \Omega (X_{i_1}, \cdots, X_{i_k}))
\]

with \( S = \{(j_1, \cdots, j_k) | 1 \leq j_1 < \cdots < j_k \leq n \text{ with } n-i+1 \leq k \leq n\} \) and \( r \) equal to the binomial coefficient

\[
\binom{k-1}{n-i}
\]

where \( \bigvee r X \) is the one point union of \( r \) copies of \( X \).

If we rename the spaces of Theorem 2, we may write \( \Omega F_i \sim \Omega \bigvee _{j=1}^N \Sigma Y_j \). This is the case studied by Hilton and Milnor. Their result shows that \( \Omega F_i \) is homotopy equivalent to a weak infinite product, \( \bigprod _{j=1}^n \Omega \Sigma Y_j \), where each \( Y_j = \Sigma^r \Lambda (Y_1^{(i)}, \cdots, Y_N^{(n)}) \) for some \((N+1)\)-tuple, \((r, i_1, \cdots, i_N)\). Since each \( Y_j, j \leq N \), is of the form \( \Sigma^{n-i-1} \Lambda ((\Omega X_1)^{(i_1)}, \cdots, (\Omega X_n)^{(n)}) \), it follows that each \( Y_j, j > N \), is of the form \( \Sigma^r \Lambda ((\Omega X_1)^{(i_1)}, \cdots, (\Omega X_n)^{(n)}) \). We thus have:

THEOREM 3. \( \Omega T_i(X_1, \cdots, X_n) \) is homotopy equivalent to a weak infinite product, \( \bigprod _{j=1}^n \Omega \Sigma X_j \), where each \( X_j, j > n \), equals

\[
\Sigma^r \Lambda ((\Omega X_1)^{(i_1)}, \cdots, (\Omega X_n)^{(n)})
\]

for some \((n+1)\)-tuple, \((r, j_1, \cdots, j_n)\), depending upon \( j \). In addition there exists an algorithm for computing the set of \((n+1)\)-tuples over which the product is taken.

In particular the algorithm is given by combining the Hilton-Milnor theorem with Theorem 2. Note that the \( X_i, i \leq n \), of Theorem 3 need not be suspensions. However, if each \( X_i = \Sigma Y_i \), for some space \( Y_i \), a further decomposition is possible as seen by the following theorem.

THEOREM 4. If \( r \geq 1 \), \( \Sigma^r \Lambda \Omega \Sigma (Y_1, \cdots, Y_m) \) is homotopy equivalent to a weak infinite product, \( \bigprod _{i=m+1}^n \Omega \Sigma Y_i \), where each \( Y_i, i \geq m+1 \), is equal to \( \Sigma^i \Lambda (Y_1^{(i)}, \cdots, Y_m^{(m)}) \) for some \((m+1)\)-tuple, \((i, i_1, \cdots, i_m)\). Moreover, an explicit algorithm can be given for computing the set of \((m+1)\)-tuples over which the product is taken.
Theorem 1 follows from Theorems 3 and 4.

The proof of Theorem 4 is modeled after [2]. The set of $Y_j$, $j > m$, of Theorem 4 is called a set of $\Lambda$-basic products and is defined inductively as follows. The basic product of weight one are $Y_1, \ldots, Y_m$, and $\Sigma^{m-1}\Lambda(Y_1, \ldots, Y_m) = Y_{m+1}$. Those of weight two are $Y_{m+1} = \Lambda(Y_{m+1}, Y_j)$, $j = 1, \ldots, m$. Define $e$ by setting $e(h) = 0$ if $1 \leq h \leq m + 1$ and $e(h) = h - (m + 1)$ if $m + 1 < h \leq 2m + 1$. Let $n > 2$. Assume inductively that the products of weight less than $n$ have been defined and are ordered and that $e(i)$ is defined for all such $i$. The basic products of weight $n$ are all elements $\Lambda(Y_i, Y_j)$ such that weight $Y_i + \text{weight } Y_j = n$ and $e(i) \leq j < i$. These are ordered arbitrarily among themselves and are greater than all products of lesser weight. Let $e(h) = j$ if $Y_h = \Lambda(Y_i, Y_j)$. This completes the inductive description of \{ $Y_j$ \}.

References


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