DIFFERENTIABLE FUNCTIONS ON CERTAIN
BANACH SPACES

BY ROBERT BONIC¹ AND JOHN FRAMPTON

Communicated by G. A. Hunt, October 21, 1964

The main result in this note, Theorem 2, can be thought of as a very strong maximum modulus type theorem. For example, let $D$ be a bounded connected open set in $\mathbb{C}(0, 1)$, and let $f: \text{Cl}D \to \mathbb{R}^n$ be continuous and differentiable in $D$. Then $f$ is determined by its values on the boundary of $D$. More exactly, $f(\text{Cl}D) \subseteq \text{Cl}f(\partial D)$. More generally, if $F$ is any Banach space and $f: \text{Cl}D \to F$ is completely continuous and differentiable in $D$, then $f(\text{Cl}D) \subseteq \text{Cl}f(\partial D)$. Note that these results are false if $C(0, 1)$ is replaced by a Hilbert space.

**Theorem 1.** Let $D$ be a connected bounded open set in $l^p$ where $p$ is not an even integer. Assume $f$ is a real-valued function, continuous on $\text{Cl}D$ and $n$-times differentiable in $D$ with $n \geq p$. Then $f(\text{Cl}D) \subseteq \text{Cl}f(\partial D)$.

This generalizes a result proved in 1954 by Kurzweil [1]. Kurzweil assumed that $f$ was $n$-times continuously differentiable, that $D$ was a ball $B(x_0, r)$, and showed that $\inf \{ ||f(x) - f(x_0)|| : ||x - x_0|| = r \} = 0$.

**Corollary 1.** Let $f$ be an $n$-times differentiable function on $l^p$, where $n \geq p$, and $p$ is not an even integer. If $f$ has its support in a bounded set, then $f$ is identically zero.

In particular, it follows that, for $n \geq p$, $C^n$ partitions of unity do not exist whenever $p$ is not an even integer. This partially settles a question raised in Lang [2]. It should be noted, however, that this is implied by Kurzweil's result.

**Corollary 2.** Let $E$ be a Banach space containing a subspace equivalent to $l^1$. Assume $D$ is a connected bounded open set in $E$, and that $f$ is a real-valued function continuous on $\text{Cl}D$ and differentiable in $D$. Then $f(\text{Cl}D) \subseteq \text{Cl}f(\partial D)$.

$C(0, 1)$ and $L^1(0, 1)$ are examples of spaces where Corollary 2 holds. More generally, any separable Banach space with an unconditional basis and nonseparable dual contains a subspace equivalent to $l^1$. It may be that any separable Banach space with a nonseparable dual has a subspace equivalent to $l^1$. Corollary 2 generalizes an unpublished result of Edward Nelson who showed that, in $C(0, 1)$, differentiable functions with bounded support are identically zero.

¹ Research supported in part by NSF grant GP-1645.

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**Theorem 2.** Let $E$ be a Banach space containing a subspace equivalent to $l^1$, let $F$ be any Banach space, and let $D$ be a bounded open connected set in $E$. Assume $f : ClD \to F$ is continuous, and that $f'(x)$ exists and is a completely continuous mapping for all $x \in D$. Then $f(ClD) \subseteq Clf(\partial D)$.

**Corollary 1.** Let $E$ and $F$ be as in the theorem and let $T : E \to F$ be completely continuous and differentiable. Then $T(ClD) \subseteq ClT(\partial D)$ for any bounded connected open set $D \subseteq E$.

This follows from the fact that if $T : E \to F$ is completely continuous and $T'(x)$ exists, then $T'(x)$ is a completely continuous linear mapping.

Letting $F$ be the reals gives the following “sup principle”.

**Corollary 2.** Let $E$ and $D$ be as in the theorem, and let $f$ be a real-valued function continuous on $ClD$ and differentiable in $D$. Then $\sup_{ClD} f(x) = \sup_{D} f(x)$.

Note that $f(x) = 1 - \|x\|^2$ shows that $E$ cannot be replaced by a Hilbert space.

**Corollary 3.** Let $M$ be a differentiable manifold modelled on $E$ where $E$ contains a subspace equivalent to $l^1$, and let $N$ be any differentiable manifold. Suppose $f : M \to N$ is differentiable and, for each $x$, $f'(x) : T_x(M) \to T_{f(x)}(N)$ is a completely continuous mapping. Let $(U, g)$ be a chart where $gU \subseteq E$ is bounded, open, and connected. Then $f(ClU) \subseteq Clf(\partial U)$.

It is well known that if $p$ is an even integer, the norm on $l^p$ is $C^\infty$, and if $p$ is not even the norm is $C^q$, where $q$ is the greatest integer strictly less than $p$. The argument in Lang [2] then shows that $C^\infty$- and $C^q$-approximation holds in these respective spaces. It follows from Theorem 1 that for $p$ not even, $(q+1)$-differentiable approximation does not hold. Restrepo [3] showed that a Banach space has an equivalent $C^1$-norm if and only if its dual space is separable. It follows that $C^1$-approximation then holds for such spaces. It follows from Theorem 2 that if $E$ is a Banach space containing a subspace equivalent to $l^1$, then not even differentiable approximation holds. In the following we show that $C^\infty$-approximation holds for $c_0$. Restrepo’s result shows that $c_0$ has an equivalent $C^1$-norm, and it is natural to ask if $c_0$ has an equivalent $C^\infty$-norm. However, we do not even know if $c_0$ has an equivalent $C^2$-norm. This result has also been observed by Edward Nelson.
Remark. $C^\infty$-approximation holds in $c_0$.

Simply let $g: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function satisfying $g(t) = 1$ for $|t| \leq 1/2$, $g(t) = 0$ for $|t| \geq 1$, and $0 < g(t) \leq 1$ for $(1/2) < |t| < 1$. Let $x = (x_1, x_2, \ldots) \in c_0$ and define $f(x) = \prod_{i=1}^\infty g(x_i)$. Then $f$ is a $C^\infty$ function nonzero in the open unit ball, and zero off it. The argument is then completed as in Lang [2].

Complete details, extensions, and applications of the results in this note will be published elsewhere.

BIBLIOGRAPHY


Cornell University and
State University of New York at Stony Brook