
$E$ is a complex locally convex vector space, in which every closed bounded subset is complete. Let $T: E \rightarrow E$ be a linear continuous operator with nonempty spectrum, possessing a continuous inverse $T^{-1}: E \rightarrow E$.

Assume the family $(T^n)_{n=1}^\infty$ to be equicontinuous.

Is it true that there is an invariant closed nontrivial linear subspace for $T$? (For a Banach space the answer is yes.) (Received December 4, 1964.)


$E$ and $F$ are Banach spaces, $F$ reflexive, $D$ is a subset of $E$ and $T: D \rightarrow F$ a nonlinear contraction, i.e., $\|Tx_1 - Tx_2\|_F \leq \|x_1 - x_2\|_E$ whenever $x_1, x_2 \in D$.

Can $T$ be extended to a contraction $\bar{T}: E \rightarrow F$? (For $E = F =$ Hilbert space the answer is yes.) (Received December 4, 1964.)


Let $D$ represent the operator $d/dx$. Consider the factorization

$$D^2 + a_1(x)D + a_2(x) = (D + b_1(x))(D + b_2(x)),$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are polynomials in $x$ of degree less than $p$, a prime, and the equality is required to hold modulo $p$. What is the number of irreducible linear differential operators for the case where $a_1(x)$ and $a_2(x)$ are required, respectively, to have degrees $m_1$ and $m_2$? Generalize to the case of linear differential operators of the form $D^n + a_1(x)D^{n-1} + \cdots + a_n(x)$. (Received November 30, 1964.)


Under what condition on the function $r(t) \geq 0$ can one assert that all solutions of $u'(t) + au(t - r(t)) = 0$ approach zero as $t \rightarrow \infty$?

Under what conditions do all solutions of $u'(t) = au(t - r(t)) = \sin bt$ approach $c \sin bt$ as $t \rightarrow \infty$?

If all solutions of $u'(t) + au(t - r) = 0$ approach zero as $t \rightarrow \infty$, and if $|r(t) - r| \leq \epsilon$ for $t \geq 0$, do all solutions of $u'(t) + au(t - r(t)) = 0$, as $t \rightarrow \infty$, for $\epsilon$ sufficiently small? (Received November 30, 1964.)

It has been recognized in recent years [cf. Good, *Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes*, Proc. Cambridge Philos. Soc. 56 (1960), 367–380] that the multidimensional Lagrange expansion is a very useful tool in many parts of analysis and mathematical physics.

Regarding the nonlinear integral equation,

\[ u(x) = f(x) + t \int_0^1 k(x, y) \phi(u(y)) \, dy, \]

as a limiting form of the simultaneous system of equations,

\[ u(x_i) = f(x_i) + t \sum_{j=1}^N w_j k(x_i, x_j) \phi(u(x_j)), \]

obtained by numerical quadrature, we can obtain an expansion of the form

\[ \psi(u) = \psi(f) + t \psi_1 + \cdots + t^n \psi_n + \cdots, \]

which is analogous to the Lagrange expansion. Can one obtain this expansion in a simpler fashion?

Generally, considering a nonlinear equation

\[ u = f + t N(u), \]

where \( N \) is a nonlinear operation, can one obtain a generalized Lagrange expansion in terms of functional derivatives? (Received December 4, 1964.)


Let \( q_1(x) \) and \( q_2(x) \) be quadratic polynomials which approach \( + \infty \) and \( x \to + \infty \). An important role in dynamic programming is played by the easily established fact that

\[ \min_y [q_1(x - y) + q_2(y)] = q_3(x), \]

where \( q_3(x) \) is again a quadratic polynomial in \( x \). Are these the only functions which display this invariance under the foregoing composition rule? Specifically,

(a) Let \( q(x, a) \) be a function of a scalar \( x \) and an \( N \)-dimensional vector \( a \). To what extent are \( q \) and \( \phi \) determined by the relation

\[ \min_y [q(x - y, a) + q(y, b)] = q(x, \phi(a, b)), \]
where \( \phi \) is an \( N \)-dimensional function of \( a \) and \( b \)?

(b) What functions allow the composition

\[
\min_{y} [q(h_1(x, y), a) + q(h_2(x, y), b)] = q(x, \phi(a, b)),
\]

under the further condition that the minimizing value is uniquely determined?

(c) What are the appropriate multidimensional versions of these problems—and solutions?

For a background discussion and some results, one can refer to Chapter X of R. Bellman and S. Dreyfus, *Applied dynamic programming*, Princeton Univ. Press, Princeton, N. J., 1962. (Received December 7, 1964.)