A NEW INVARIANT OF HOMOTOPY TYPE
AND SOME DIVERSE APPLICATIONS

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Let $X$ be a connected, locally finite simplicial polyhedron. Let $X^X$ be the space of maps from $X$ to $X$ with the compact-open topology. Let $x_0 \in X$ be taken as a base point in $X$, then the evaluation map $p: X^X \to X$ defined by $p(f) = f(x_0)$ for $f \in X^X$ is continuous. Now $p$ induces the homomorphism

$$p_* : \pi_1(X^X, 1_x) \to \pi_1(X, x_0),$$

where $1_x \in X^X$ is the identity map. Hence $p_*\pi_1(X^X, 1_x)$ is a subgroup of the fundamental group of $(X, x_0)$.

**Proposition 1.** $p_*\pi_1(X^X, 1_x)$ considered as a subgroup of $\pi_1(X, x_0)$ is an invariant of homotopy type.

In [2], this invariant is studied and theorems are obtained which bear on the study of $X^X$, groups of homeomorphisms, homological group theory and knot theory. Most of these results come from the following theorem.

**Theorem 2.** Let $X$ have the homotopy type of a compact, connected polyhedron with nonzero Euler-Poincaré number. Then $p_*\pi_1(X^X, 1_x) = 0$.

The proof of this employs Nielsen-Wecken fixed-point class theory ([1] and [5]).

Let $G(X)$ be the group of homeomorphisms of a manifold $X$, and let $G_0(X)$ be the isotropy group over $x_0$. Then there is an exact sequence [3]

$$\cdots \to \pi_i(G_0(X), 1_x) \xrightarrow{i_*} \pi_i(G(X), 1_x) \xrightarrow{p'_*} \pi_i(X, x_0) \to \cdots,$$

where $p': G(X) \to X$ is the evaluation map.

**Corollary 3.** Let $X$ be as in Theorem 2. Then $p'_* \pi_1(G(X), 1_x) = 0$. In particular, if $\pi_2(X, x_0) = 0$, then $i_* : \pi_1(G_0(X), 1_x) \cong \pi_1(G(X), 1_x)$.

This follows because $p'_* \pi_1(G(X), 1_x) \subseteq p_* \pi_1(X^X, 1_x)$.

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Theorem 4. If $X$ is an aspherical polyhedron, then $p_*\pi_1(X^x, 1_x) = Z(\pi_1(X, x_0))$, the center of $\pi_1(X, x_0)$.

Theorems 2 and 4 combine to give us the following corollaries:

Corollary 5. If $X$ has the same homotopy type as a compact, connected, aspherical polyhedron with nonzero Euler-Poincaré number, then $Z(\pi_1(X, x_0)) = 0$.

John Stallings, in [4], has put this result in a purely algebraic setting; namely, if a group $G$ admits a finite resolution, then, if $Z(G)$ is nontrivial, the (suitably defined) Euler-Poincaré number is zero.

Alexander’s Duality and the last corollary gives us a result suggested by L. P. Neuwirth.

Corollary 6. Suppose that $X$ is a subcomplex of the $n$-sphere $S^n$ whose Euler characteristic is different from that of $S^n$. If $S^n - X$ is connected and aspherical, then $\pi_1(S^n - X)$ has no center.

Finally, we are able to show the following:

Theorem 7. If $X$ is aspherical, then

$$\pi_1(X^x, 1_x) \cong Z(\pi_1(X, x_0)),$$

$$\pi_n(X^x, 1_x) \cong 0, \quad n > 1.$$

Note that Theorem 7 and Theorem 2 give us:

Corollary 8. If $X$ has the homotopy type of an aspherical compact polyhedron whose Euler characteristic is different from zero, then the identity component of $X^x$ is contractible.

BIBLIOGRAPHY


Institute for Defense Analyses and University of Illinois