TOPOLOGY OF QUATERNIONIC MANIFOLDS

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We give here a quaternionic analogue (Theorem 4) of the Hodge decomposition theorem [2, p. 26] for a Riemannian manifold with holonomy group contained in Sp(n) × Sp(1). Applying Chern's theorem in [1] (also [3]), we obtain some consequences on Betti numbers (Theorem 5).

Let $K^n$ denote the $n$-dimensional vector space over the field $K$ of quaternions, with the inner product

$$(p, q) = \frac{1}{2} \sum_{i=1}^{n} (p_i q_i + q_i p_i),$$

where

$$p = (p_1, \cdots, p_n), \quad q = (q_1, \cdots, q_n)$$

are quaternions. Let $Sp(n)$ be the set of all endomorphisms, $A$, of $K^n$, satisfying the identity $(Ap, Aq) = (p, q)$. $Sp(n)$ is the set of all $n \times n$ matrices preserving the inner product. Then $Sp(1)$ is the set of all unit quaternions. We define the action of $Sp(n) \times Sp(1)$ on $K^n$ as follows:

$$(A, \lambda)p = Ap\lambda, \quad \text{for } (A, \lambda) \in Sp(n) \times Sp(1),$$

i.e., we multiply $p$ on the left by the matrix $A$ and on the right by the unit quaternion $\lambda$.

DEFINITION. We define three skew symmetric bilinear forms $\Omega_I$, $\Omega_J$ and $\Omega_K$ on $K^n$ as follows:

$$\Omega_I(p, q) = (pi, q),$$

$$\Omega_J(p, q) = (pj, q) \quad \text{and}$$

$$\Omega_K(p, q) = (pk, q).$$

Note that $\Omega_I$, $\Omega_J$ and $\Omega_K$ may be thought of as exterior 2-forms of $K^n$ considered as a $4n$-dimensional real vector space.

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**Definition.** We define an exterior 4-form $\Omega$ on $\mathbb{K}^n$ by

$$\Omega = \Omega_I \wedge \Omega_J + \Omega_J \wedge \Omega_K + \Omega_K \wedge \Omega_L.$$  

**Theorem 1.** $\Omega$ is invariant under the action of $\text{Sp}(n) \times \text{Sp}(1)$.

**Theorem 2.** $\Omega^n = \Omega \wedge \Omega \wedge \cdots \wedge \Omega$ ($n$ times) $\neq 0$.

**Definition.** A $4n$-dimensional Riemannian manifold $M$ is called a quaternionic manifold if its holonomy group is a subgroup of $\text{Sp}(n) \times \text{Sp}(1)$.

If $M$ is a quaternionic manifold of dimension $4n$, then, by Theorems 1 and 2, we have a differential 4-form $\Omega$ on $M$ of maximal rank (i.e., $\Omega^n \neq 0$) which is parallel. Hence, $\Omega$ is a harmonic form. From the fact that $\Omega^n \neq 0$, we have

**Theorem 3.** If $B_i$ denotes the $i$th Betti number of a quaternionic manifold $M$ of dimension $4n$, then we have $B_{4i} \neq 0$ for $i = 0, 1, \ldots, n$.

We define the operator $*$ which sends a $p$-form into a $(4n - p)$-form in the usual way.

**Definition.** Define two operators $L$ and $\Lambda$ on the differential forms by

$$Lw = \Omega \wedge w, \quad \Lambda w = *(\Omega \wedge *w).$$

A differential form $w$ is called effective if $\Lambda w = 0$.

**Theorem 4.** Let $w$ be a $p$-form; then

$$w = w_e^p + Lw_e^{p-4} + \cdots + Lw_e^{p-4r}, \quad \text{for } p \leq n,$$

where $w_e^k$ is an effective $k$-form, and $r = [p/4]$.

From Theorem 4, it follows that $L$ sending $p$-forms into $(p+4)$-forms is 1-1 for $p \leq n-4$.

**Theorem 5.** We have an increasing sequence of Betti numbers,

$$B_i \leq B_{i+4} \leq \cdots \leq B_{i+4r}, \quad \text{for } i + 4r \leq n, \quad i = 0, 1, 2 \text{ or } 3.$$

**Bibliography**


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