FUNCTIONS WITH THE HUYGENS PROPERTY

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A $C^2$ function $u(x, t)$ belongs to class $H$, for $a<t<b$, and is called a generalized temperature function, if and only if it is a solution of the generalized heat equation

$$\Delta_x u(x, t) = (\partial/\partial t) u(x, t),$$

where $\Delta_x f(x) = f''(x) + (2\nu/x)f'(x)$, $\nu$ a fixed positive number. The fundamental solution of this equation is

$$G(x, y; t) = (\sqrt{2t}) I_i(xy/2t) \exp[-(x^2 + y^2)/4t],$$

with $I_\nu(z) = c_\nu z^{\nu+1/2} - i\nu 2^{-\nu} I_{\nu - 1/2}(z)$, and $I_\gamma(z)$ the Bessel function of order $\gamma$ of imaginary argument. We write $G(x; t)$ for $G(x, 0; t)$. The function $u(x, t)$ is said to have the Huygens property, that is, it belongs to class $H^*$, for $a<t<b$, if and only if $u(x, t) \in H$ there, and

$$u(x, t) = \int_0^\infty G(x, y; t - t') u(y, t') \, d\mu(y), \quad d\mu(x) = (1/c_\nu)x^{2\nu} \, dx,$$

for every $t$, $t'$, $a<t'<t<b$, the integral converging absolutely. A generalized heat polynomial $P_{n,v}(x, t)$ is defined by

$$P_{n,v}(x, t) = \sum_{k=0}^n 2^{2k} \binom{n}{k} \frac{\Gamma(\nu + \frac{1}{2} + n)\Gamma(\nu + \frac{1}{2} + n - k)}{\Gamma(\nu + \frac{1}{2} + n - k)} x^{2n-2k}\sqrt{t},$$

and its Appell transform $W_{n,v}(x, t)$ is given by

$$W_{n,v}(x, t) = G(x, t)P_{n,v}(x/t, -1/t).$$

The object of this paper is to summarize the principal results derived in characterizing a generalized temperature function which may be represented either by the series expansion $\sum_{n=0}^\infty a_n P_{n,v}(x, t)$ or by $\sum_{n=0}^\infty b_n W_{n,v}(x, t)$, with convergence taken in the $L^2$, as well as in the pointwise, sense. Details and proofs will appear later. The work is an extension of the theory developed by Rosenbloom and Widder in [3]. Some of the preliminary results for this study were also de-

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rived by F. M. Cholewinski, and it has been called to our attention that Louis Bragg has done work in this area.

The region of convergence of the series \( \sum_{n=0}^{\infty} a_n P_n(x, t) \) is, in general, a strip \( |t| < \sigma \), whereas that of the series \( \sum_{n=0}^{\infty} b_n W_n(x, t) \) is a half plane \( t > \sigma \geq 0 \). Indeed, we have

**THEOREM 1.** If \( \lim_{n \to \infty} |a_n|^{1/n4n/e} = 1/\sigma < \infty \), then the series

\[
\sum_{n=0}^{\infty} a_n P_n(x, t)
\]

converges absolutely in the strip \( |t| < \sigma \) and does not converge everywhere in any including strip.

**THEOREM 2.** If \( \lim_{n \to \infty} |b_n|^{1/n4n/e} = \sigma < \infty \), then the series

\[
\sum_{n=0}^{\infty} b_n W_n(x, t)
\]

converges absolutely in the half plane \( t > \sigma \geq 0 \) and does not converge everywhere in any including half plane.

Within their regions of convergence, \( \sum_{n=0}^{\infty} a_n P_n(x, t) \) and \( \sum_{n=0}^{\infty} b_n W_n(x, t) \) each defines a generalized temperature function \( u(x, t) \), with the additional fact, in the first case, that \( u(x, 0) \) is an even entire function of growth \((1, 1/4\sigma)\). An entire function \( \phi(x) = \sum_{n=0}^{\infty} c_n x^n \) is said to be of growth \((\rho, \tau)\) if and only if \( \lim_{n \to \infty} n|c_n|^{\rho/n} \leq \tau e^\rho \).

Since we find that \( u(x, t) \) has an expansion \( \sum_{n=0}^{\infty} a_n P_n(x, t) \) in the largest strip \( |t| < \sigma \) for which \( u(x, t) \in H^* \), we note that the role of membership in class \( H^* \) in expansions in terms of generalized heat polynomials is analogous to that of analyticity in expansions in Taylor series. This is established in the following.

**THEOREM 3.** A necessary and sufficient condition that

\[
u(x, t) = \sum_{n=0}^{\infty} a_n P_n(x, t),
\]

the series converging for \( |t| < \sigma \), is that \( u(x, t) \in H^* \) there. The coefficients \( a_n \) have either of the determinations

\[
a_n = u^{(2n)}(0, 0)/(2n)!, \quad \text{or}
\]

\[
a_n = \{ \Gamma(\nu + 1/2)/[2^{4n} \pi^{3/2} \Gamma(\nu + 3/2 + n)] \} \int_{0}^{\infty} u(y, -t) W_n(y, t) \, d\mu(y),
\]

\( 0 < t < \sigma \).
In addition, the following result gives us a complex determination of the coefficients.

**Theorem 4.** If \( u(x, t) = \sum_{n=0}^{\infty} a_n P_n(x, t) \), the series converging for \(|t| < \sigma\), then

\[
a_n = \left\{ (-1)^n \Gamma(n + \frac{1}{2})/[2^{4n+1} \Gamma(n + \frac{3}{2})] \right\} \int_0^\infty u(ix, t) W_{n+1}(x, t) \, d\mu(x),
\]

\[0 < t < \sigma.\]

Membership in class \( H^* \) is not sufficient for an expansion in terms of \( W_n(x, t) \), as is indicated by the function \( u(x, t) = 1 \), which is in \( H^* \) for \(-\infty < x < \infty\) but cannot be represented by the expansion \( \sum_{n=0}^{\infty} b_n W_n(x, t) \). Instead we have the following modification of the dual to Theorem 3.

**Theorem 5.** A necessary and sufficient condition that

\[
u(x, t) = \sum_{n=0}^{\infty} b_n W_{n+1}(x, t),
\]

the series converging for \( t > \sigma \geq 0 \), is that \( u(x, t) \in H^* \) there and that

\[
\int_0^\infty |u(x, t)| e^{2\sigma/t} \, d\mu(x) < \infty, \quad \sigma < t < \infty.
\]

The coefficients \( b_n \) have the determination

\[
b_n = \left\{ \Gamma(n + \frac{1}{2})/[2^{4n+1} \Gamma(n + \frac{3}{2})] \right\} \int_0^\infty u(y, -t) W_{n+1}(y, -t) \, d\mu(y),
\]

\[\sigma < t < \infty.\]

The proof of this theorem depends on one which establishes necessary and sufficient conditions of a different nature for such an expansion.

**Theorem 6.** A necessary and sufficient condition that

\[
u(x, t) = \sum_{n=0}^{\infty} b_n W_{n+1}(x, t),
\]

the series converging for \( t > \sigma \geq 0 \), is that

\[
u(x, t) = \int_0^\infty g(xu) e^{-tu^2} \phi(y) \, d\mu(y), \quad \sigma < t < \infty,
\]

where \( g(z) = 2^{(1-s)\Gamma(1/2)} z^{1/2} \Gamma(s) J_{1/2}(z) \), \( \phi(y) \) is an even entire function of growth \((1, \sigma)\), and \( b_n = \phi(2n)(0)/[(2n)!(-4)^n] \).
For expansions in the $L^2$ sense, we have the following:

**Theorem 7.** If $u(x, t) \in H^\ast$, $-\sigma \leq t < 0$, and if $u(x; t) \left[ G(x; -t) \right]^{1/2} \in L^2$, for each fixed $t$, $-\sigma \leq t < 0$, $0 \leq x < \infty$, then, for $-\sigma \leq t < 0$,

$$\lim_{N \to \infty} \int_0^\infty G(x, -t) \left| u(x, t) - \sum_{n=0}^N a_n P_{n,v}(x, t) \right|^2 \, d\mu(x) = 0,$$

and

$$\int_0^\infty G(x, -t) \left| u(x, t) \right|^2 \, d\mu(x) = \sum_{n=0}^\infty 2^{4n} n! \Gamma(n + \frac{1}{2} + n) \left| a_n \right|^2 \gamma^{2n} / \Gamma(n + \frac{1}{2}),$$

where

$$a_n = \left\{ \Gamma(n + \frac{1}{2}) \left[ 2^{4n} n! \Gamma(n + \frac{1}{2} + n) \right] \right\} \int_0^\infty u(y, t) W_{n,v}(y, -t) \, d\mu(y),$$

$$0 \leq n < \sigma.$$

The example $u(x, t) = e^{ax^2} g(ax)$ illustrates a limitation of this theorem. Although, in this case, $u(x, t) \in H^\ast$ for $0 < t < \infty$, as well as for $-\infty < t \leq 0$, so that it may be represented by $\sum_{n=0}^\infty a_n P_{n,v}(x, t)$, with convergence in the pointwise sense for $0 < t < \infty$, Theorem 7 fails to give such an expansion in the $L^2$ sense for $0 < t < \infty$, since $u(x, t) \left[ G(x, -t) \right]^{1/2} \not\in L^2$ for $0 < t < \infty$. Thus, for $t > 0$, we need an additional result.

**Theorem 8.** If $u(x, t) \in H^\ast$, $0 < t \leq \sigma$, and if $u(ix, t) \left[ G(x, t) \right]^{1/2} \in L^2$, for each fixed $t$, $0 < t \leq \sigma$, $0 \leq x < \infty$, then, for $0 < t \leq \sigma$,

$$\lim_{N \to \infty} \int_0^\infty G(x, t) \left| u(ix, t) - \sum_{n=0}^N a_n P_{n,v}(x, -t) \right|^2 \, d\mu(x) = 0,$$

and

$$\int_0^\infty G(x, t) \left| u(ix, t) \right|^2 \, d\mu(x) = \sum_{n=0}^\infty 2^{4n} n! \Gamma(n + \frac{1}{2} + n) \left| a_n \right|^2 \gamma^{2n} / \Gamma(n + \frac{1}{2}),$$

where

$$a_n = \left\{ \Gamma(n + \frac{1}{2}) \left[ 2^{4n} n! \Gamma(n + \frac{1}{2} + n) \right] \right\} \int_0^\infty u(ix, t) W_{n,v}(x, t) \, d\mu(x),$$

$$0 < t \leq \sigma.$$
Theorem 9. If $u(x, t) \in H^*$, $t \geq \sigma > 0$, and if $u(x, t) [G(ix, t)]^{1/2} \in L^2$, for each fixed $t$, $t \geq \sigma > 0$, $0 \leq x < \infty$, then for $t \geq \sigma > 0$,

$$\lim_{N \to \infty} \int_0^\infty G(ix, t) \left| u(x, t) - \sum_{n=0}^N b_n W_{n, \sigma}(x, t) \right|^2 d\mu(x) = 0,$$

and

$$\int_0^\infty G(ix, t) \left| u(x, t) \right|^2 d\mu(x) = \sum_{n=0}^\infty 2^{4n-2n-1} n! \Gamma(\nu + \frac{1}{2} + n) \left| a_n \right|^2 \frac{t^{2n-3n-1}}{\Gamma(\nu + \frac{1}{2})},$$

where

$$b_n = \frac{\Gamma(\nu + \frac{1}{2})}{[2^{2n} n! \Gamma(\nu + \frac{1}{2} + n)]} \int_0^\infty u(x, t) P_{n, \sigma}(x, -t) d\mu(x),$$

$$0 \leq t < \infty.$$

The properties of $P_{n, \sigma}(x, t)$ and $W_{n, \sigma}(x, t)$ play a central role in the development of the theory. Of primary importance is the fact that the polynomials $P_{n, \sigma}(x, t)$ and the functions $W_{n, \sigma}(x, t)$ form a bi-orthogonal system in the sense that

$$\int_0^\infty W_{n, \sigma}(x, t) P_{m, \sigma}(x, -t) d\mu(x) = \delta_{mn} 2^{4n} n! \Gamma(\nu + \frac{1}{2} + n) / \Gamma(\nu + \frac{1}{2}).$$

In addition, the equation

$$G(x, y; s - t) = \sum_{n=0}^\infty \frac{\Gamma(\nu + \frac{1}{2})}{[2^{2n} n! \Gamma(\nu + \frac{1}{2} + n)]} W_{n, \sigma}(y, s) P_{n, \sigma}(x, -t)$$

is fundamental. We make repeated use, in the proofs, of asymptotic estimates of $P_{n, \sigma}(x, t)$ and $W_{n, \sigma}(x, t)$. Indeed, these estimates, in addition to the fact that $P_{n, \sigma}(x, t) \in H^*$, for $-\infty < t < \infty$, and $W_{n, \sigma}(x, t) \in H^*$, for $0 < t < \infty$, enable us to prove that the integral determinations of the coefficients in the above series expansions are all independent of $t$.

References