RESEARCH ANNOUNCEMENTS

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THEOREM OF THE THREE CLOSED GEODESICS

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1. The theorem we are referring to states that on every compact riemannian manifold \( M \) there exist three simple closed geodesics of which the energy is bounded in terms of a map of a sphere into \( M \), cf. §6 for a precise formulation; there we also explain why this is the best possible general solution of the problem to give a lower bound for the number of simple closed geodesics of bounded energy on a compact riemannian manifold.

We call a geodesic simple if it does not cover another geodesic.

The theorem is a simultaneous generalization of the theorem of Lusternik-Schnirelmann [4] that on a surface of the type of the 2-sphere \( S^2 \) there are three simple geodesics, and of the theorem of Fet [2] that on every compact riemannian manifold \( M \) there is one simple closed geodesic.

We obtain this theorem, and several other new results on the existence of closed geodesics, from a refinement of the previously employed methods for studying the space of closed curves and singling out among these the geodesics. For a brief historical survey we refer to our note [3].

2. Our approach uses substantially the Morse theory on infinite dimensional manifolds as developed recently by Palais and Smale [7], [8], [9]. In our case, the infinite dimensional manifold is the space \( \Lambda(M) \) of absolutely continuous maps \( f = (f(t)) \) of the parametrized circle \( S^1 = [0, 1]/\{0, 1\} \) into \( M \) which have, in any local chart, square integrable derivatives.

\( \Lambda(M) \) is called the space of parametrized closed curves on \( M \). It is a manifold modeled after a separable Hilbert space, cf. Palais [7]. The homotopy type of \( \Lambda(M) \) depends only on the homotopy type of \( M \).

The riemannian metric \( \langle \cdot, \cdot \rangle \) on \( M \) determines on \( \Lambda(M) \) the energy

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function by $E(f) = \frac{1}{2}\int_0^1 (f'(t), f'(t))\,dt$ and the riemannian metric $\ll \cdot, \cdot \rr$ by
\[
\ll X, X \rr = \int_0^1 \left( \langle X(t), X(t) \rangle + \langle DX(t)/dt, DX(t)/dt \rangle \right)dt
\]
where $X = (X(t))$ is a tangent vector at a point $f = (f(t))$ of $\Lambda(M)$.

$E$ is differentiable. $\Lambda(M)$ is complete with respect to the metric $\ll \cdot, \cdot \rr$. With the help of this metric we define on $\Lambda(M)$ the vector field $-\text{grad } E$. The condition (C) of Palais and Smale is satisfied which allows to develop the Morse theory of the function $E$ on $\Lambda(M)$, cf. [7] and [9].

A point $f \in \Lambda(M)$ is critical with respect to $E$, i.e., $\text{grad } E(f) = 0$, if and only if $f$ is a geodesic, parametrized proportional to arc length. Note that the point curves on $M$ are critical points of $\Lambda(M)$. They form a nondegenerate critical submanifold $\Lambda^0(M)$ of index 0 which is isomorphic to $M$.

3. As additional feature we have on $\Lambda = \Lambda(M)$ the left action of the orthogonal group $O(2)$ which is induced from the usual action of $O(2)$ on the circle $S^1$. This action is continuous, it leaves the energy function $E$ invariant and it is isometric with respect to $\ll \cdot, \cdot \rr$. It follows that also the vector field $-\text{grad } E$ is invariant under the action of $O(2)$.

We introduce the quotient space $\Pi(M) = \Lambda(M)/O(2)$. Let $\pi$ be the quotient map $\Lambda(M) \to \Pi(M)$. $\Pi = \Pi(M)$ is called the space of unparametrized closed curves on $M$. The homotopy type of $\Pi(M)$ depends only on the homotopy type of $M$.

Put $\pi \Lambda^0 = \Pi^0$; this is isomorphic to $\Lambda^0$.

Let $\phi_t: \Lambda(M) \to \Lambda(M) (t \geq 0)$ be the one-parameter transformation semigroup induced from the integration of the vector field $-\text{grad } E$. $\phi_t$ carries orbits of $O(2)$ into orbits. Hence, it induces a one-parameter transformation semigroup $\psi_t: \Pi(M) \to \Pi(M)$, satisfying $\psi_t \circ \pi = \pi \circ \phi_t$.

4. When talking about homology and cohomology, we always mean the singular theory with $\mathbb{Z}_2$ coefficients.

Let $u$ be a singular cycle of $\Pi \mod \Pi^0$. Then $c(u) = \lim_{t \to \infty} \max_{x \in u} \psi_t(u)$ exists. Let $z$ be a homology class of $\Pi \mod \Pi^0$. Then also $c(z) = \inf_{u \in z} c(u)$ exists. $c(z)$ is called the critical value of $z$.

There exists a critical point on $\Lambda(M)$ of energy $c(z) > 0$ if $z \neq 0$. Hence, by taking the underlying simple closed geodesic, we find that for $z \neq 0$ there exists a simple closed geodesic of energy $\leq c(z)$ on $M$.

There arises the question when two different homology classes $z$ and $z'$ in this way give rise to two different simple closed geodesics. To formulate a criterion we introduce the following concepts:
A homology class \( z = z_k \in H_k(\Pi, \Pi^0) \) is called subordinated to the class \( z' = z_{k+l} \in H_{k+l}(\Pi, \Pi^0) \) if both are nonzero and \( l > 0 \) and if there is a cohomology class \( \xi \in H^l(\Pi - \Pi^0) \) such that \( z_k = z_{k+l} \cap \xi \). Here we use the fact that there is a natural pairing \( H_\ast(\Pi, \Pi^0) \otimes H^\ast(\Pi - \Pi^0) \to H_\ast(\Pi, \Pi^0) \) given by the cap product. The concept of subordination was introduced in this connection by Lyusternik [5] and used also by Al'ber [1].

Let \( u = \{ b: (K, K^0) \to (\Pi, \Pi^0) \} \) be a singular cycle of \( \Pi \mod \Pi^0, K \) being a complex. We say that \( u \) has local cross sections in \( \Lambda \) if every point of \( K - K^0 \) is contained in an open set \( U \) of \( K \) for which there exists an equivariant map \( a_\mathbb{U}: \mathbb{U} \times O(2) \to \Lambda \) which induces \( b|_\mathbb{U} \).

Then we may state as the fundamental result of this paper the following

**Lemma.** Let \( z \) and \( z' \) be homology classes of \( \Pi(\mathcal{M}) \mod \Pi^0(\mathcal{M}) \) such that \( z \) is subordinated to \( z' \) and \( z \) as well as \( z' \) can be represented by cycles possessing local cross sections in \( \Lambda(\mathcal{M}) \). Then there exist on \( \mathcal{M} \) two simple closed geodesics of energy \( \leq c(z') \).

Hence, we get as many simple closed geodesics on \( \mathcal{M} \) as we can find pairwise subordinated homology classes in \( \Pi(\mathcal{M}) \mod \Pi^0(\mathcal{M}) \) which can be represented by cycles possessing local cross sections in \( \Lambda(\mathcal{M}) \).

5. Let \( S \) be an irreducible compact symmetric space of rank 1, i.e., a sphere or a projective space. As was indicated already in [3], we have in \( \Pi = \Pi(S) \) the subspace \( C = C(S) \) of circles. Let \( C^0 = C^0(S) \) be the subspace of point circles, isomorphic to \( S \). Then we noted in [3] that the inclusion \( i: C \mod C^0 \to \Pi \mod \Pi^0 \) is injective in homology, and the same is true also for the inclusion \( i: C - C^0 \to \Pi - \Pi^0 \). Note that the cycles of \( \Pi(S) \) which lie in \( C(S) \) possess local cross sections in \( \Lambda(S) \).

One can now show that if \( h: S \to \mathcal{M} \) is a homotopy equivalence, then there exist maps \( H_\ast(\Pi(S), \Pi^0(S)) \to H_\ast(\Pi(\mathcal{M}), \Pi^0(\mathcal{M})) \) and \( H_\ast(\Pi(S) - \Pi^0(S)) \to H_\ast(\Pi(\mathcal{M}) - \Pi^0(\mathcal{M})) \) which are bijective and which carry cycles lying in the space of circles \( C(S) \) into cycles having local cross sections in \( \Lambda(M) \).

So all that remains to be done for finding a lower bound for the number of simple closed geodesics on a space \( \mathcal{M} \) of the homotopy type of \( S \) is to determine the maximal number of pairwise subordinated classes on the space of circles on \( S \). This amounts to computing the cup length of the ring \( H^\ast(C(S) - C^0(S)) = H^\ast(G(S)) \) where \( G(S) \) is the space of great circles on \( S \).

The results are:
THEOREM 1. On a compact riemannian manifold $M = M^n$ of the homotopy type of the sphere $S^n$ there exist $2n - s - 1$ simple closed geodesies, with $0 \leq s = n - 2^2 < 2^h$. The energy of these geodesies is bounded in terms of a homotopy equivalence $h: S^n \to M$.

Recall that the other compact irreducible symmetric spaces of rank 1 are the projective spaces $P^n(\lambda)$ over the reals ($\lambda = 1$), the complex numbers ($\lambda = 2$) or the Cayley numbers ($\lambda = 8$), of real dimension $n = m\lambda \geq 2^\lambda$ and $n = 2\lambda = 16$ for $\lambda = 8$.

THEOREM 2. On a compact riemannian manifold $M = M^n$ of the homotopy type of the projective space $P^n(\lambda)$, $n = m\lambda$, there exist at least $2n - (2\lambda - 1)s - 1$ simple closed geodesies, with $0 \leq s = m - 2^\lambda < 2^h$. The energy of these geodesies is bounded in terms of a homotopy equivalence $h: P^n(\lambda) \to M$.

6. THEOREM 3. On a compact riemannian manifold $M = M^n$ there exist at least three simple closed geodesies. For $\pi_1(M) = 0$, the energy of these geodesies is bounded in terms of a map of a $k$-sphere into $M$, $2 \leq k \leq n$.

REMARK. This is the best possible general result, as is shown by the following example due to Morse [6]. Given an arbitrarily large (and not too small) real number $c$, there exists a 2-dimensional ellipsoid $E^2$ with three different axes, all having their length close to 1, such that the only simple closed geodesics on $E^2$ of energy $\leq c$ are the three principal ellipses. But these are just the three simple closed geodesics which are obtained, in the manner described below, from the Gauss map $h: S^2 \to E^2$.

For an indication of the proof we restrict ourselves to the case $\pi_1(M) = 0$; the case $\pi_1(M) \neq 0$ is reduced to this case, if the universal covering $\tilde{M}$ of $M$ is compact, otherwise one works with elements in the fundamental group $\pi_1(M)$.

For $\pi_1(M) = 0$ there exists a $k$, $2 \leq k \leq n$, and a map $h: S^k \to M$ giving a nontrivial homology class in $H_k(M) = H_k(M, \mathbb{Z}_2)$. This induces a map $h_*: H_\ast(C(S^k), C^0(S^k)) \to H_\ast(\Pi(M), \Pi^0(M))$ which is injective for three pairwise subordinated homology classes

$y_{i(k-1)} \in H_{i(k-1)}(C(S^k), C^0(S^k)), \quad i = 1, 2, 3,$

and which has the property that their images can be represented by cycles possessing local cross sections in $\Lambda(M)$. So we can apply the Lemma.

We give a description of the classes $y_{i(k-1)}$: First note that $C(S^k) - C^0(S^k)$ is an open $(k-1)$-disc bundle over the space $G(S^k)$ of great
circles on $S^k$ where $G(S^k)$ is isomorphic to the grassmannian $G(2, k-1)$ of 2-planes in real $(k+1)$-space. Let $w^{k-1} \in H^{k-1}(G(2, k-1))$ be the Whitney class of this bundle and $u^{k-1} \in H^{k-1}(C(S^k), C^0(S^k))$ the Thom class of the Thom space $C(S^k)$ mod $C^0(S^k)$ of the bundle. One finds that $(u^{k-1})^2 = u^{k-1} \cup (w^{k-1})^2 \in H^{2k-2}(C(S^k), C^0(S^k))$ is $\neq 0$. Let $y_{2k-3}$ be the generator of $H_{2k-3}(C(S^k), C^0(S^k))$, which is dual to $(u^{k-1})^2$, and put $y_{2k-2} = y_{2k-3} \wedge w^{k-1}$ and $y_{k-1} = y_{2k-2} \wedge w^{k-1}$.

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