A factor is a ring of operators whose center consists only of scalar multiples of the identity. Murray and von Neumann have defined various kinds of factors, calling a continuous factor with finite trace a type II$_1$ factor [3], [4]. Dixmier began the detailed study of maximal abelian subalgebras of type II$_1$ factors. He defined regular, semi-regular but not regular, and singular maximal abelian subalgebras, and showed that at least one of each type exists [2]. His II$_1$ factors turn out to be hyperfinite in algebraic type. The factors we consider are also hyperfinite. In this note we discuss their semi-regular subalgebras, and present an isomorphism invariant which allows us to obtain new existence results.

Let $\mathfrak{A}$ be a hyperfinite factor, $\mathcal{R}$ a maximal abelian subalgebra of $\mathfrak{A}$. For any subring $D$ of $\mathfrak{A}$, $N(D)$ is the ring generated by all unitaries which leave $D$ invariant, and $N^k(D) = N[N^{k-1}(D)]$. In particular, we let $N(\mathcal{R}) = \mathfrak{R}$. $\mathcal{R}$ is semi-regular but not regular iff $\mathfrak{R}$ is a factor not equal to $\mathfrak{A}$. In [5] we defined an isomorphism invariant for such subalgebras, which we called length. If $\mathcal{R} \subset \mathcal{P} \subset N(\mathcal{P}) \subset \cdots \subset N^L(\mathcal{P}) = \mathfrak{A}$, (when $\mathcal{R} \neq \mathcal{P} \neq N(\mathcal{P}) \neq \cdots \neq N^L(\mathcal{P})$) then $L$ is the length of $\mathcal{R}$. By constructing a semi-regular subalgebra $\mathcal{R}$ of every length $L = 1, 2, 3, \ldots$, we obtained an infinite sequence of subalgebras which could not be pairwise connected by $*$-automorphisms of $\mathfrak{A}$.

Another possible invariant is product type. Suppose $\mathcal{R}$ has length $L$. Then $\mathcal{R}$ is of product type $\alpha$, $0 \leq \alpha \leq L$, iff the following holds: For every $t$, $1 \leq t \leq \alpha$, there exist $S_1$ and $S_2$ in $N^{t-1}(\mathcal{P}) \cap N^t(\mathcal{P})$ such that the product $S_1S_2 \neq 0$ is in $N^{t-1}(\mathcal{P}) \cap N^t(\mathcal{P})$. But for $s$ such that $\alpha \leq s \leq L$, every $T_1$ and $T_2$ in $N^{s-1}(\mathcal{P}) \cap N^s(\mathcal{P})$ have their product $T_1T_2$ in $N^{s-1}(\mathcal{P})$. (Taking of orthogonal complements is meaningful, for within a II$_1$ factor, the weak, strong, and Hilbert space (metric) closures of a subalgebra all coincide [4]. The metric topology is based on the norm derived from the scalar product $(A, B) = \text{Tr}(B^*A)$ for $A, B$ in $\mathfrak{A}$.)

**Theorem 1.** Suppose $\mathcal{R}$ and $\mathcal{R}'$ are semi-regular but not regular subalgebras of $\mathfrak{A}$, and $\mathcal{R}$ has product type $\alpha$, while $\mathcal{R}'$ has product type $\beta$.

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1 This work was done as part of NSF Research Participation for College Teachers, the University of Oklahoma, summer, 1964.
\( \alpha', \alpha \neq \alpha' \). Then there does not exist a \(*\)-automorphism \( \Theta \) of \( \mathfrak{A} \) such that \( \Theta (R') = R \).

**Proof.** We can assume \( \alpha' > \alpha \), so that \( \alpha' \geq \alpha + 1 \). Letting \( t = \alpha + 1 \) in the definition of product type, we know we can choose \( S_1, S_2 \) in \( N^\alpha(P') \cap N^{\alpha+1}(P') \) such that \( S_1 S_2 = S_3 \neq 0 \) is in this set also. Suppose there exists \( \Theta \) such that \( \Theta (R') = R \). By a standard argument, it follows that \( \Theta [N(R')] = N(R) \) or \( N(P') = P \), and inductively, \( \Theta [N^\alpha(P')] = N^\alpha(P) \) and \( \Theta [N^{\alpha+1}(P')] = N^{\alpha+1}(P) \). Let \( \Theta (S_i) = T_i \), so that \( T_i \in N^{\alpha+1}(P) \) for \( i = 1, 2, \) or \( 3 \). Now if \( A \in N^{\alpha}(P') \), then \( (S_i, A) = 0 \) = \( \text{Tr}(A * S_i) = \text{Tr}[\Theta(A * S_i)] \) (since the trace function is unique) = \( \text{Tr}[(\Theta(A) * \Theta(S_i)) = (\Theta(S_i), \Theta(A))] = (T_i, \Theta(A)) = 0 \). As \( A \) ranges over \( N^\alpha(P') \), \( \Theta(A) \) takes on all values in \( N^\alpha(P) \), so we must have \( T_i \in N^\alpha(P) \). Thus \( T_i \) is in \( N^\alpha(P) \cap N^{\alpha+1}(P) \) for \( i = 1, 2, \) or \( 3 \).

Now \( \alpha + 1 > \alpha \), so letting \( s = \alpha + 1 \) and considering the product type of \( \mathfrak{A} \), it follows that \( T_1 T_2 \) is in \( N^s(P) \). But \( T_1 T_2 = \Theta(S_i) \Theta(S_2) = \Theta(S_1 S_2) \) = \( \Theta(S_3) = T_3 \). Since \( T_3 \) is in \( N^s(P) \), this leads to a contradiction. \( (T_3 \neq 0 \) since \( S_3 \neq 0 \) and \( \Theta \) is an automorphism.) Therefore we cannot have \( \Theta (R') = R \).

**Theorem 2.** There are \( (L+1) \) semi-regular subalgebras of length \( L \) which cannot be pairwise connected by \(*\)-automorphisms of \( \mathfrak{A} \). Specifically, these have product types \( \alpha = 0, 1, 2, \ldots, L \).

We give an indication of the proof, which is constructive and depends on the results of [5]. For each \( n = 1, 2, 3, \ldots \), the matrix units (of all the \( 2^p \) by \( 2^p \) matrix algebras, where \( p \) is an odd multiple of \( n \)) are divided into \( n \) orthogonal sets. These are called \( \mathfrak{S}_0, \mathfrak{S}_1, \ldots, \mathfrak{S}_n \), and the set \( \mathfrak{C}_k = \bigcup_{n=0}^{n} \mathfrak{S}_n \). The ring \( R(\mathfrak{C}_k) \) is defined as the weak closure of the algebra generated by matrix units in \( \mathfrak{C}_k \). Then for each \( n \) and for \( 0 \leq \alpha \leq n - \alpha \), we construct \( R_n(\alpha) \), a semi-regular subalgebra. The chain for \( R_n(\alpha) \) is such that \( N^t(P_n(\alpha)) = R(\mathfrak{C}_n) \) for \( 0 \leq \alpha \leq n \) and \( N^t(P_n(\alpha)) = R(\mathfrak{C}_{a+\alpha}) \) for \( \alpha \leq s \leq n - \alpha \). Since \( N^{n-\alpha}(P_n(\alpha)) = R(\mathfrak{C}_n) = \mathfrak{A} \), we have \( L = n - \alpha \).

But these properties are sufficient to show that \( R_n(\alpha) \) has product type \( \alpha \). So for any \( L = 1, 2, 3, \ldots \), we can take \( \alpha = 0, 1, \ldots, L \), and \( n = \alpha + L \). Then, by Theorem 1, there does not exist an \(*\)-automorphism \( \Theta \) of \( \mathfrak{A} \) such that \( \Theta (R_{\alpha+L}(\alpha')) = R_{\alpha+L}(\alpha') \) when \( \alpha \neq \alpha' \).

A generalization of the concept of product type permits one to construct \( 2^L \) nonisomorphic semi-regular maximal abelian subalgebras of every length \( L \). However, the construction becomes extremely involved.
BIBLIOGRAPHY


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PURE SUBGROUPS HAVING PRESCRIBED SOCLES

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Let $B = \sum B_n$ be a direct sum of cyclic groups where, for each positive integer $n$, $B_n = \sum C(p^n)$ is zero or homogeneous of degree $p^n$ where $p$ is a fixed prime. Denote by $\bar{B}$ the torsion completion of $B$ in the $p$-adic topology. Following established terminology [1], we refer to $\bar{B}$ as the closed primary groups with basic subgroup $B$.

A primary group $G$ is said to be pure-complete if each subsocle of $G$ supports a pure subgroup of $G$. A semi-complete group was defined by Kolettis in [6] to be a primary group which is the direct sum of a closed group and a direct sum of cyclic groups.

For a particular $B$, I exhibited in [3] nonisomorphic pure subgroups $H$ and $K$ of $\bar{B}$ having the same socle. Using this example, Megibben [7] was the first to show the existence of a primary group without elements of infinite height which is not pure-complete. We mention that each semi-complete group is pure-complete [4]. The purpose of this note is to announce the following theorem and corollaries; proofs will appear in another paper.

**THEOREM.** Suppose that $B$ is unbounded and countable and that $S$ is any proper dense subsocle of $\bar{B}$ such that $|S| = 2^{\aleph_0}$. Then $S$ supports more than $2^{\aleph_0}$ pure subgroups of $\bar{B}$ which are isomorphically distinct.

The theorem has the following implications.

**COROLLARY 1.** Suppose that $B$ is unbounded and countable and that