A CONJECTURE OF J. NAGATA ON DIMENSION AND METRIZATION

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THEOREM 1. A metrizable space $X$ is of dimension $\leq n$ if and only if $X$ admits a metric compatible with the topology which satisfies the condition $(n)$: For any $n+3$ points $x, y_1, \ldots, y_{n+2}$ of $X$ there exist distinct indices $i, j$ such that $d(y_i, y_j) \leq d(x, y_i)$.

In this paper we outline briefly a proof of Theorem 1, which was conjectured by J. Nagata [1].

By dimension we shall always mean covering dimension. A family of subsets of $X$ is discrete if each point of $X$ has a neighborhood which meets at most one member of the family. For a subset $A$ of $X$ and a family $\mathcal{C}$ of subsets of $X$, let $S(A, \mathcal{C})$ denote the union of $A$ and all those $C \in \mathcal{C}$ such that $C \cap A \neq \emptyset$. For each integer $n \geq 0$, let

$$S^n(A, \mathcal{C}) = \begin{cases} A & \text{if } n = 0, \\ S(S^{n-1}(A, \mathcal{C}), \mathcal{C}) & \text{if } n > 0; \end{cases}$$

$[\mathcal{C}]^n = \{S^n(C, \mathcal{C}) : C \in \mathcal{C}\}$.

Let $X$ be a metrizable space of dimension $\leq n$. For each positive integer $j$ there exist $n+1$ discrete families of open sets, $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{n+1}$ such that if $\mathcal{U}_j = \bigcup_{i=1}^{n+1} \mathcal{U}_i$, then:

1. each $\mathcal{U}_j$ covers $X$;
2. for each $x \in X$, $\{S(x, \mathcal{U}_j) : j = 1, 2, \ldots\}$ is a neighborhood base at $x$;
3. $[\mathcal{U}_{j+1}]^{n+1}$ refines $\mathcal{U}_j$ for each $j$;
4. if $j < k$ and $1 \leq i \leq n+1$, each member of $[\mathcal{U}_k]^{n+1}$ meets at most one member of $\mathcal{U}_j$.

The $\mathcal{U}_j$ are defined inductively on $j$. Their construction relies on a new characterization of dimension [2].

THEOREM 2. A metrizable space $X$ is of dimension $\leq n$ if and only if for each open cover $\mathcal{C}$ of $X$ there exist $n+1$ discrete families of open sets, $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{n+1}$ such that $\bigcup_{i=1}^{n+1} \mathcal{U}_i$ is a cover of $X$ which refines $\mathcal{C}$.

PROOF OF THEOREM 1. Let $\mathbb{R}^*$ denote the set of dyadic rationals in the open interval $(0, 1)$. For each $m \in \mathbb{R}^*$ there exist $n+1$ discrete

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families of open sets $S_m^1, S_m^2, \ldots, S_m^{n+1}$ such that if $S_m = \bigcup_{i=1}^{n+1} S_m^i$ then:

1. each $S_m$ covers $X$;
2. for each $x \in X$, $\{S(x, S_m) : m \in R^*\}$ is a neighborhood base at $x$;
3. if $m < p \in R^*$, then $S_m$ refines $S_p$;
4. if $m, p \in R^*$, and $j$ is a positive integer such that $2^{-j} \leq m < p \leq 2^{-(j-1)}$, then $S_m^i$ refines $S_p^i$ for each $1 \leq i \leq n+1$;
5. if $m < p \in R^*$, $1 \leq i \leq n+1$ and $U \in S_m^i$, $V \in S_p^i$, then either $U \subseteq V$ or $U \cap V = \emptyset$;
6. if $m, p \in R^*$, $m + p < 1$, and if $U \in S_m$, $V \in S_p$ are such that $U \cap V \neq \emptyset$, then there exists $W \in S_{m+p}$ such that $U \cup V \subseteq W$.

The $S_m$ are constructed from the $U_i^j$ as follows: Let

$$*U_j^i = \{S^{(i)}(U, U_{j+1}^i) : U \in U_j^i\}; \quad *U_j = \bigcup_{i=1}^{n+1} *U_j^i.$$

For $A \subseteq X$, $1 \leq i \leq n+1$, $j \geq 1$ and $k \geq 0$, let

$$T^k(A, i, j) = \begin{cases} S(A, *U_j^i) & \text{if } k = 0, \\ S(T^{k-1}(A, i, j), *U_{j+k}^i) & \text{if } k > 0; \end{cases}$$

$$T(A, i, j) = \bigcup_{k=0}^{\infty} T^k(A, i, j).$$

For $m \in R^*$ of the form $m = \sum_{i=0}^{k} 2^{-m_i}$, where $1 \leq m_1 < m_2 < \cdots < m_k$, let $\bar{m} = \sum_{i=1}^{k} 2^{-m_i}$. Further, for $A \subseteq X$ and $1 \leq i \leq n+1$, let

$$i \Delta m = \begin{cases} T(A, i, m_1 + 1) & \text{if } t = 1, \\ T(S^t(i \Delta \bar{m}, *U_m), i, m_1) & \text{if } t > 1; \end{cases}$$

$$S_m^i = \{i \Delta m : U \in U_m^i\}.$$

This complicated construction is necessary to achieve condition (SO). A simpler construction in which (SO) does not hold was used by Nagata [1] to obtain a weaker result.

Define a non-negative real-valued function $d$ on $X \times X$ by

$$d(x, y) = \begin{cases} 1 & \text{if } y \in S(x, S_m) \text{ for any } m \in R^*, \\ \inf\{m \in R^* : y \in S(x, S_m)\} & \text{if } y \in S(x, S_m) \text{ for some } m \in R^*. \end{cases}$$

$d$ is clearly symmetric. It follows from (3') that $d(x, y) = 0$ only if $x = y$. The triangular inequality follows from (6'). Thus $d$ is a metric on $X$. By (1') and (2') $d$ is compatible with the topology of $X$. If $x, y_1, \ldots, y_{n+2}$ are points of $X$, then by (4') and (SO) there exist indices $i, j, i \neq j$, such that $d(y_i, y_j) \leq d(x, y_j)$.

Conversely, suppose that $X$ is a metric space with metric $d$ satisfy-
ing condition (n). Let $C$ be an open cover of $X$. It is easily shown by use of Zorn's Lemma that if $A \subseteq X$ and $\epsilon > 0$, there is a subset $B$ of $X$ which is maximal (under inclusion) with respect to the properties:

(i) $B \cap A = \emptyset$;
(ii) $\{S_\epsilon(x) : x \in B\}$ refines $C$; $(S_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\})$
(iii) if $x \neq y \in B$, then $d(x, y) \geq \epsilon$.

Hence we may construct subsets $A_i$ of $X$ for $i = 1, 2, \cdots$, inductively on $i$, which are maximal with respect to the properties:

(1) $A_i \cap \{y \in X : \text{for some } 1 \leq j < i \text{ and } x \in A_j, d(x, y) < 2^{-i}\} = \emptyset$;
(2) $\{S_{2^{-i}}(x) : x \in A_i\}$ refines $C$;
(3) if $x \neq y \in A_i$, then $d(x, y) \geq 2^{-i}$.

Let $\mathcal{U} = \{S_{2^{-i}}(x) : x \in A_i ; i = 1, 2, \cdots\}$. The maximality of the $A_i$ insures that $\mathcal{U}$ covers $X$. Conditions (1) and (3) and the condition on the metric insure that $\mathcal{U}$ is of order $\leq n + 1$. Thus the proof is complete.

**References**


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