GENERALIZED UNITARY OPERATORS

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1. Let $\mathbb{C}$ be the complex field and $\Gamma$ be the unit circle \( \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \). For a non-negative integer $m$ or for $m = \infty$, let $C^m(\Gamma)$ be the space of all $m$-times continuously differentiable functions on $\Gamma$. (Here we consider $\Gamma$ as a $C^\infty$-manifold in the natural way. Thus, any \( f \in C^m(\Gamma) \) can be identified with an $m$-times continuously differentiable periodic function $f(\theta)$ of a real variable $\theta$ with period $2\pi$.) $C^m(\Gamma)$ is an algebra as well as a Banach space if $m$ is finite, a Fréchet space if $m = \infty$, with the usual sup-norms for derivatives.

We shall say that a mapping $\gamma$ of $\Gamma$ into $\mathbb{C}$ is a $C^m$-curve if $\gamma$ can be extended onto a neighborhood $V$ of $\Gamma$ (the extended map will also be denoted by $\gamma$) in such a way that it is one-to-one on $V$ and $\gamma$ and $\gamma^{-1}$ are both $m$-times continuously differentiable (as functions in two variables) on $V$ and $\gamma(V)$ respectively.

Let $E$ be a Hausdorff locally convex space over $\mathbb{C}$ such that the space $\mathcal{L}(E)$ of all continuous linear operators on $E$ endowed with the bounded convergence topology is quasi-complete.

2. $C^m(\gamma)$-operators.

DEFINITION. Let $\gamma$ be a $C^m$-curve. $T \in \mathcal{L}(E)$ is called a $C^m(\gamma)$-operator if there exists a continuous algebra homomorphism $W$ of $C^m(\Gamma)$ into $\mathcal{L}(E)$ such that $W(1) = I$ and $W(\gamma) = T$. If $\gamma$ is the identity map: $\gamma(\theta) = e^{i\theta}$, then a $C^m(\gamma)$-operator is called a $C^m$-unitary operator. ( Cf. Kantrovitz' approach in [1].)

THEOREM 1. If $T$ is a $C^m(\gamma)$-operator, then $\text{Sp}(T) \subseteq \gamma(\Gamma)$.

If $H$ is a Hilbert space, $T \in \mathcal{L}(H)$ is a $C^0$-unitary operator if and only if it is similar to a unitary operator on $H$. In this sense, $C^m$-unitary operators on $E$ generalize the notion of unitary operators on a Hilbert space.

The homomorphism $W$ in the above definition is uniquely determined by $T$ and $\gamma$. Thus, we call $W$ the $C^m(\gamma)$-representation for $T$. The uniqueness can be derived from the following approximation theorem: Given a $C^m$-curve $\gamma$, let $\lambda_0$ be a point inside the Jordan curve

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2 $\text{Sp}(T)$ is the spectrum of $T$ in Waelbroeck's sense. See [2] for the definition.
\( \gamma(\Gamma) \). Then the set \( \{(P \circ \gamma)/(\gamma - \lambda_0)^n; P: \text{polynomial in one complex variable, } n: \text{integer } \geq 0\} \) is dense in \( C^m(\Gamma) \).

**Theorem 2.** Let \( T \) be a \( C^m(\gamma) \)-operator for a \( C^m \)-curve \( \gamma \) and let \( W \) be the \( C^m(\gamma) \)-representation for \( T \).

(i) If \( A : L(E) \) commutes with \( T \), then \( A \) commutes with each \( W(f) \), \( f \in C^m(\Gamma) \).

(ii) If \( F \) is a closed subspace of \( E \) left invariant under \( T \) and \( (\lambda_0 I - T)^{-1} \) for some \( \lambda_0 \) inside of \( \gamma(\Gamma) \), then it is left invariant under any \( W(f) \), \( f \in C^m(\Gamma) \).

**3. Characterization theorem.** We recall ([2] and [4]) that \( S \in L(E) \) with compact spectrum is called a \( C^m \)-scalar operator if there exists a continuous homomorphism \( U \) of the topological algebra \( C^m(\mathbb{R}^2) \equiv C^m(\mathbb{C}) \) into \( L(E) \) such that \( U(1) = I \) and \( U(\lambda) = S^\lambda \). In this case, the support of \( U \) is contained in \( \text{Sp}(S) \).

Now, we consider the following statements concerning \( S \in L(E) \), depending on \( m \) and a \( C^m \)-curve \( \gamma \):

- \( I_n(m) : S \) is a \( C^m(\gamma) \)-operator.
- \( II_n(m) : S \) is a \( C^m \)-scalar operator such that \( \text{Sp}(S) \subseteq \gamma(\Gamma) \).
- \( III_n(m) : S^{-1} \in L(E) \) and for each continuous semi-norm \( q \) on \( L(E) \), there exist a non-negative integer \( m_q (= m, \text{if } m \text{ is finite}) \) and \( M_q > 0 \) such that
  
  \[
  q(S^k) \leq M_q |k|^{m_q} \quad \text{for all } k = \pm 1, \pm 2, \ldots .
  \]

(Cf. [1].)

- \( IV_n(m) : \text{Sp}(S) \subseteq \gamma(\Gamma) \) and for each continuous semi-norm \( q \) on \( L(E) \), there exist a non-negative integer \( m_q (= m, \text{if } m \text{ is finite}) \) and \( M_q > 0 \) such that
  
  \[
  q(R_\lambda) \leq M_q d_\lambda^{-m_q - 1} \quad \text{for all } \lambda \text{ with } 0 < d_\lambda < 1,
  \]

where \( R_\lambda = (\lambda I - S)^{-1} \) for \( \lambda \in \text{Sp}(S) \) and \( d_\lambda = \text{dis}(\lambda, \text{Sp}(S)) \). (Cf. [6].)

When \( \gamma \) is the identity map, we omit the subscript \( \gamma \) in the notations \( I_n(m) \), \( II_n(m) \) and \( IV_n(m) \); in particular,

\( I(m) : S \) is a \( C^m \)-unitary operator.

**Theorem 3 (The Characterization Theorem).**

(i) \( I(m) \Rightarrow II(m) \Rightarrow III(m) \Rightarrow IV(m) \Rightarrow I(m + 2) \). *In particular, \( I(\infty) \), \( II(\infty) \), \( III(\infty) \) and \( IV(\infty) \) are mutually equivalent.*

(ii) \( I_\gamma(m) \Rightarrow II_\gamma(m) \Rightarrow IV_\gamma(m) \Rightarrow I_\gamma(m + 2) \).

\(^8\) \( C^m \) is the space of all \( m \)-times continuously differential functions on \( \mathbb{R}^2 = \mathbb{C} \).

The topology in it is defined by sup. of derivatives on compact sets.

\(^4\) \( \lambda \) denotes the identity function \( f(\lambda) = \lambda \).

\(^5\) In the implication \( IV \gamma (m) \Rightarrow I \gamma (m + 2) \), we are assuming that \( \gamma \) is a \( C^{m+2} \)-curve.
In particular, \( I_i(\infty) \), \( II_j(\infty) \) and \( IV_j(\infty) \) are mutually equivalent.

4. Here, we shall give indications of proofs of Theorem 3, (i). The proofs of (ii) are similar but more complicated.

\( I(m) \Rightarrow II(m) \): If \( W \) is the \( C^m(e^{it}) \)-representation for \( S \), then we define \( U(\phi) = W(\phi(e^{it})) \) for \( \phi \in C^m \). Then \( U \) is a \( C^m \)-representation for \( S \).

\( II(m) \Rightarrow III(m) \): Since \( S^k = U(\lambda^k) \), we obtain (1) evaluating the norms of \( \lambda^k \) on neighborhoods of \( \Gamma \) and using the continuity of \( U \).

\( III(m) \Rightarrow IV(m) \): If \( |\lambda| < 1 \), then \( R_\lambda = -\sum_{k=0}^{\infty} \lambda^k S^{-(k+1)} \); if \( |\lambda| > 1 \), then \( R_\lambda = \sum_{k=0}^{\infty} \lambda^{-(k+1)} S^k \). Hence, (2) follows from (1).

\( IV(m) \Rightarrow I(m+2) \): For \( f \in C^{m+2}(\Gamma) \), we define

\[
W(f) = \lim_{\epsilon \to 0+} \frac{1}{2\pi} \left\{ \int_0^{2\pi} f(\theta) [R_{(1+\epsilon)e^{i\theta}} - R_{(1-\epsilon)e^{i\theta}}] e^{i\theta} d\theta \right\}.
\]

By a method due to Tillmann ([5] and [6]), we see that the right-hand side is well-defined and that \( W \) is the \( C^{m+2}(e^{it}) \)-representation for \( S \).

5. Corollary and examples.

**Corollary.** If \( S_i(i = 1, 2) \) is a \( C^{m_i} \)-unitary operator and if \( S_1 \) and \( S_2 \) commute, then \( S_1 S_2 \) is a \( C^{m_1 + m_2} \)-unitary operator.

This is a consequence of Theorem 2, Theorem 3 and the corollary to Proposition 3.1 of [3].

**Examples.** Let \( \mathcal{S}(\mathbb{R}^n) \) be the Fréchet space of rapidly decreasing functions on \( \mathbb{R}^n \). \( \mathcal{S}'(\mathbb{R}^n) \)' is the space of tempered distributions. Let \( E = \mathcal{S}(\mathbb{R}^n) \) or \( \mathcal{S}'(\mathbb{R}^n) \)' The translations \( \tau_\alpha : [\tau_\alpha f](x) = f(x + \alpha) \) are \( C^\infty \)-unitary operators on \( E \); the Fourier transform is a \( C^2 \)-unitary operator on \( E \).

**References**


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