The object of this paper is to gain some information about the unstable piecewise linear groups. The tool that we use for this purpose is the s-cobordism theorem (which has been established for piecewise linear manifolds by J. Stallings [9] and D. Barden [1]). All manifolds and micro-bundles in this paper are piecewise linear, unless otherwise specified.

**Theorem 1.** Let $M^m$ be a compact manifold such that $\pi_1(\partial M) \cong \pi_1 M$ by inclusion, and let $f: K^k \to M^m$ be a simple homotopy equivalence of a finite simplicial $k$-complex with $M$. Then if $m \geq 6$, $m \geq 2k + 1$, there is a compact manifold $L$ such that $\pi_1(\partial L) \cong \pi_1 L$, and $L \times I \cong M$. If $m \geq 7$, $m \geq 2k + 2$, $L$ is uniquely determined.

**Proof.** We first observe that the pair $(M, \partial M)$ is $(m - k - 1)$-connected. Indeed, since inclusion induces an isomorphism of fundamental groups, we can use the relative Hurewicz theorem to compute the first nonvanishing relative homotopy group: $\pi_i(M, \partial M) \cong \pi_i(\tilde{M}, \partial \tilde{M})$ (where $\tilde{M}$ denotes the universal cover), $\pi_i(\tilde{M}, \partial \tilde{M}) \cong H_i(\tilde{M}, \partial \tilde{M}) \cong H_{n-i}(\tilde{M})$, by duality, where $c$ denotes compact cohomology, and $H_{n-i}(\tilde{M}) \cong H_{n-i}(\tilde{K})$ vanishes for $i < m - k$.

It follows that for $m \geq 2k + 1$, $f$ is homotopic to a map $g: K \to \partial M$. If, now $m \geq 2k + 2$ we can move $g$ into general position (see e.g. [11, Chapter 6, Theorem 18]) and so suppose it an imbedding. Take a regular neighbourhood $L$ of $g(K)$ in $\partial M$. Then $L$ is a manifold, and the inclusion $L \subset M$ is a simple homotopy equivalence.

If $m = 2k + 1$, $g$ will in general have singularities, transverse self-intersections of $k$-simplexes of $K$. For each such selfintersection $Q = g(P_1) = g(P_2)$, we join $P_1$ to $P_2$ by a path $\alpha$ in $K$ such that $g(\alpha)$ is a nullhomotopic loop (since $g_*: \pi_1(K) \to \pi_1(\partial M)$ is onto, this is possible). As $k \geq 3$, we can now map a disc $D^2$ into $\partial M$, with its interior imbedded, and meeting $g(K)$ only in its boundary, which is attached along $g(\alpha)$. Proceeding thus for each selfintersection $Q$, we obtain an imbedding of a complex $K'$ simply homotopy-equivalent to $K$; we can then take a regular neighbourhood to obtain $L$, as above. Note in either case that as regular neighbourhood of a subcomplex $K'$ of codimension $\geq 3$, $L$ has the property $\pi_1(\partial L) \cong \pi_1 L$, for $\partial L$ is a deformation retract of $L - K'$.
Take a collar neighbourhood $\partial L \times I$ of $L$ in $\partial M$ (this is possible since $L$ is a submanifold); let $L^1$ be the closure of the complement of $L \cup (\partial L \times I)$ in $\partial M$. We regard $M$ as a cobordism of $L$ and $L^1$: along the 'edge', $\partial L \times I$ is a product cobordism of $\partial L$ and $\partial L^1$. Also, the inclusion of $L$ in $M$ is a simple homotopy equivalence. To show that $M$ is an $s$-cobordism, it remains only to check that the inclusion $L^1 \subset M$ induces an isomorphism of $\pi_1$. Now the complement of $L^1$ in $\partial M$ is a regular neighbourhood of a $k$-complex, which has codimension $\geq 3$, so $\pi_1(L^1) \cong \pi_1(\partial M)$; and by hypothesis, $\pi_1(\partial M) \cong \pi_1(M)$. Hence $M$ is an $s$-cobordism which along the edge is a product cobordism; by the $s$-cobordism theorem, $M$ is a product: $M \cong L \times I$.

For the proof of uniqueness, we first show that $L$ is in any case the regular neighbourhood of a $k$-complex, given $m \geq k+4$, $m \geq 6$. For by assumption $\pi_1(\partial L) = \pi_1(L)$; now, as in the proof of existence, $(L, \partial L)$ is $(m-k-2)$-connected. By [10, Theorem 5.5], if $k \geq 2$, $L$ has a handle decomposition based on $\partial L$ with no $i$-handles for $i \leq m-k-2$; the dual decomposition has no $j$-handles for $j > k$, and so $L$ collapses onto a $k$-dimensional spine. If $k = 1$, since $m \geq 6$ we can imbed $K$ in $L$ by a simple homotopy equivalence and take a regular neighbourhood $L^1$ of the image; by [10, Theorem 6.4] (a variant of the $s$-cobordism theorem) $L$ is diffeomorphic to $L^1$. We observe that the arguments of [10] can be justified for PL-manifolds by using results from [1] or [9]; we could also use a PL version of the nonstable neighbourhood theorem of Mazur [7, p. 54].

Suppose then $M = L_1 \times I \cong L_2 \times I$, and consider the image of $L_2 \times 0$ in $\partial M$. Since $\partial M$ has dimension $\geq 2k+1$, we can deform this to be disjoint from $L_1 \times 1$, and then a further deformation puts it in the interior of $L_1 \times 0$. Write $H$ for the closure of $(L_1 \times 0) - (L_2 \times 0)$; we assert that $H$ is an $s$-cobordism, and hence a product $\partial L_2 \times I$, so that $L_1$ is homeomorphic to $L_2$. This can be proved algebraically, or we can use a direct argument by cancellation of handles: for details see Wall [10, Theorem 6.4].

We now consider piecewise linear micro-bundles. The basic information on these is contained in Milnor [8]. We write $e^r$ for the trivial micro-bundle with fibre $R^r$.

**Corollary 1.1.** For any micro-bundle $\xi^r$ over $K^k$, we can write $\xi^r + e^{2k} \cong e^r + \eta^{2k}$ for a suitable micro-bundle $\eta^{2k}$ of fibre dimension $2k$. (If $k = 2$, replace $2k$ by 5.)

**Proof.** First suppose $k \geq 3$. Then, as in the theorem, we can imbed some complex simple homotopy-equivalent to $K$ in $R^{2k}$; thicken it, and call the result $L$. The tangent micro-bundle of $L$ is $e^{2k}$. Take the
total space $M^1$ of the micro-bundle induced over $L$ by $\xi^r$, and let $M$ be a regular neighbourhood of $L$ in $M^1$: this has tangent micro-bundle $\xi^r + e^{2k}$. Now by iterating the theorem, we can write $M = N^{2k} \times D^r$, so if $\eta^{2k}$ is the tangent micro-bundle of $N^{2k}$, the result follows.

If $k = 2$, we replace $R^4$ by $R^6$, so $L$ has dimension 5. The argument concludes as before.

**Corollary 1.2.** Suppose $\xi^r$ and $\eta^r$ are stably equivalent micro-bundles over $K^k$. Then $\xi^r + e^{2k} \equiv \eta^r + e^{2k}$. (If $k = 2$, replace $2k$ by $5$.)

**Proof.** Construct $L$ as above; take regular neighbourhoods $X$ and $Y$ of $L$ in the micro-bundles induced over $L$ by $\xi$ and $\eta$. Since $\xi$ and $\eta$ are stably equivalent, for some $s$, $X \times D^s \equiv Y \times D^s$. Since $X$ and $Y$ have dimension $2k + r \geq 2k + 1$, it follows by iterating the uniqueness part of Theorem 1 that $X$ and $Y$ are PL-homeomorphic. Hence their tangent micro-bundles $\xi^r + e^{2k}$, $\eta^r + e^{2k}$ are equivalent.

**Remark 1.** To classify micro-bundles over a 1-complex, it is sufficient to be able to do it over a circle; for this we only need $\pi_0(PL_m)$, which is well known to be $\mathbb{Z}_2$. Thus if $k = 1$, we have $\xi^r = e^{r-1} + \eta^1$, and stably equivalent micro-bundles are equivalent.

**Theorem 2.** Suppose $K^k$ a compact unbounded piecewise linear submanifold of $M^m$. Then if $m \geq 3k$, $K^k$ has a piecewise linear normal microbundle in $M^m$.

**Proof.** First assume $k \geq 3$. According to Milnor [8, Theorem 4], for some $n$, $K^k$ has a normal micro-bundle $\xi^* \in M^m \times R^n$. By the above corollary, write $\xi^* + e^{2k} = \epsilon^* + \eta^{2k}$; let $N_1$ be a regular neighbourhood of $K$ in the total space of $\eta^* + e^{m-2k}$, $N_2$ a regular neighbourhood of $K$ in $M$. Then $N_1 \times D^n$, $N_2 \times D^n$ are both regular neighbourhoods of $K$ in $M \times R^n$, hence are PL-homeomorphic.

By Theorem 1, if $m \geq 6$, $N_1$ and $N_2$ are PL-homeomorphic. We assert that there is even a PL-homeomorphism inducing the identity on the subcomplex $K$. Granted this, $K$ has a normal micro-bundle in $N_1$, hence also in $N_2$, and so in $M$.

Write $i_1: K \to N_1$, $i_2: K \to N_2$ for the inclusions, and $f: N_1 \to N_2$ for the PL-homeomorphism constructed above. Then, by the construction of $f$, $f i_1 \equiv i_2$. Since $\dim N_2 = 3k \geq 2k + 2$, homotopic imbeddings are isotopic. By the covering isotopy theorem of Hudson and Zeeman [5], since (as $k \geq 2$) the codimension is $\geq 3$, we can cover the isotopy of $K$ in $N$ by an isotopy $h_t$ of $N$. Hence $h_t f i_1 = i_2$. The homeomorphism $h_t f$ now has the required properties.

In low dimensions we can use a different argument. For if $k \leq 7$, it follows from smoothing theory (see e.g. [3]) that $N_2$ and $K$ admit
compatible differential structures; if also $2m \geq 3k + 3$, by [2, Theorem 2a] the imbedding of $K$ in $N$ can be approximated by a differentiable imbedding; if finally $2m \geq 3k + 4$ these two imbeddings, being homotopic, are PL-isotopic by a theorem of Hudson [4]. Hence $M^m$ can be regarded as a smooth manifold with $K^k$ as smooth submanifold; as such it has a normal vector bundle and hence a normal PL-microbundle, according to [6, Theorem 3.2].

**Addendum to Theorem 2.** The result also holds if $k \leq 7, 2m \geq 3k + 4$.

This includes those cases of the theorem which were not covered by our first argument.

**Remark 2.** The necessity of suspending $\xi$ in the corollaries to Theorem 1—as also the lack of a uniqueness clause in Theorem 2—all stem from our inability, given a complex $K$ and PL-micro-bundle $\xi'$ over $K$, to construct a manifold $M'$ and homotopy equivalence $h: M' \rightarrow K$, such that $h^*\xi$ is equivalent to the tangent micro-bundle of $M$. (However large $r$ is, we cannot yet do this.)

Added in proof. Haefliger and the author have now proved a stability theorem for PL-micro-bundles fully analogous to the stability properties of vector bundles, and deduced that Theorem 2 holds for $m \geq 2k$.

**References**