A SIMPLE PROOF OF THE RABIN-KEISLER THEOREM

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For terminology and notation we refer to the two relevant papers of Rabin [3] and Keisler [1]. The following theorem is proved in [1] and is an improvement of the main result of [3].

THEOREM (RABIN-KEISLER). Let \( \alpha \) be an infinite nonmeasurable cardinal. Then every model of power \( \alpha \) has a proper elementary extension of the same power if and only if \( \alpha = \alpha^\omega \).

The simple proof referred to in the title does not require the elaborate apparatus of limit ultrapowers (see [1]) or the generalized continuum hypothesis and that \( \alpha \) be accessible (see [3]). On the other hand, the proof owes much to certain ideas in [3] and Keisler [2].

One direction of the theorem follows easily from elementary properties of ultrapowers. The following lemma will establish the other direction.

**Lemma.** Suppose \( \alpha \) is an infinite nonmeasurable cardinal, \( \mathcal{M} = \langle A, R, S, \ldots \rangle \) is the complete model over a set \( A \) of power \( \alpha \), and \( \mathcal{M}' = \langle A', R', S', \ldots \rangle \) is a proper elementary extension of \( \mathcal{M} \). Then \( |A'| \geq \alpha^\omega \).

**Proof.** By a well-known result in set theory (using finite sequences of elements from \( A \)), there exists a family

\[
P = \{ P_\beta : \beta < \alpha^\omega \}
\]

of countably infinite subsets \( P_\beta \) of \( A \) such that \( |P| = \alpha^\omega \) and \( P_\beta \cap P_\gamma \) is finite whenever \( \beta \neq \gamma \). Well-order each \( P_\beta \),

\[
P_\beta = \{ p_\beta n : n < \omega \}.
\]

Let \( x \in A' - A \), and let

\[
D = \{ Q : Q \subset A \text{ and } x \in Q' \}.
\]

It is easily seen that \( D \) is a nonprincipal ultrafilter over \( A \). By hypothesis \( D \) is countably incomplete. Hence, there exists a strictly decreasing sequence

\[
A = Q_0 \supset Q_1 \supset \cdots \supset Q_n \supset \cdots
\]

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of sets \( Q_n \subseteq D \) such that \( \cap_n Q_n = 0 \). Fix \( \beta \prec \alpha^w \). Define a function \( F_\beta \) mapping \( A \) onto \( P_\beta \) as follows: for each \( a \in A \),

\[
F_\beta(a) = p_{\beta n} \text{ if and only if } a \in Q_n - Q_{n+1}.
\]

Notice that the function \( F_\beta \) (considered as a binary relation) and the sets \( P_\beta, Q_n \) are among the relations listed in \( \mathfrak{M} \). Since \( \mathfrak{M} \prec \mathfrak{M}' \), it follows that \( F'_\beta \) is a function mapping \( A' \) onto \( P'_\beta \). Furthermore, for each \( a' \in A' \),

\[
F'_\beta(a') = p_{\beta n} \text{ if and only if } a' \in Q'_n - Q'_{n+1}.
\]

Since \( x \in Q_n' \) for all \( n \), we have

\[
F'_\beta(x) \in P'_\beta - P_\beta.
\]

Using the fact that \( P_\beta \cap P_\gamma \) is finite whenever \( \beta \neq \gamma \), we have \( (P_\beta \cap P_\gamma) = P'_\beta \cap P'_\gamma = P_\beta \cap P_\gamma \). Hence

\[
F'_\beta(x) \neq F'_\gamma(x), \text{ whenever } \beta \neq \gamma.
\]

So \( |A'| \geq \alpha^w \) and the lemma is proved.

References