A COMBINATORIAL THEOREM FOR
STOCHASTIC PROCESSES

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Let \( \{X(t), 0 \leq t \leq t\} \) be a stochastic process where \( t \) is a finite positive number. We associate a stochastic process \( \{X^*(u), 0 \leq u < \infty\} \) with \( \{X(u), 0 \leq u \leq t\} \) as follows: \( X^*(u) = X(u) \) for \( 0 \leq u \leq t \) and \( X^*(t+u) = X^*(t) + X^*(u) \) for \( u > 0 \). If the finite dimensional distributions of \( \{X^*(v+u) - X^*(u), 0 \leq u \leq t\} \) are independent of \( v \) for \( v \geq 0 \), then the process \( \{X(u), 0 \leq u \leq t\} \) is said to have cyclically interchangeable increments. In particular, if \( \{X(u), 0 \leq u \leq t\} \) has stationary, independent increments, and \( P\{X(0) = 0\} = 1 \), then it belongs to this class.

**Theorem.** If \( \{X(u), 0 \leq u \leq t\} \) is a separable stochastic process with cyclically interchangeable increments and if almost all sample functions are nondecreasing step functions which vanish at \( u = 0 \), then

\[
P\{X(u) \leq u \text{ for } 0 \leq u \leq t \mid X(t)\}
\]

\[
= \begin{cases} 
1 - \frac{X(t)}{t} & \text{if } 0 \leq X(t) \leq t, \\
0 & \text{otherwise},
\end{cases}
\]

with probability 1.

**Proof.** Let \( X^*(u), 0 \leq u < \infty \), be a nondecreasing step function (nonrandom) for which \( X^*(0) = 0 \) and \( X^*(t+u) = X^*(t) + X^*(u) \) if \( u > 0 \) where \( t \) is a fixed positive number. For \( u \geq 0 \) define

\[
\xi(u) = \begin{cases} 
1 & \text{if } X^*(v) - X^*(u) \leq v - u \text{ for } v \geq u, \\
0 & \text{otherwise}.
\end{cases}
\]

Obviously \( \xi(u+t) = \xi(u) \) for all \( u \geq 0 \). Now we shall prove that

\[
\int_0^t \xi(u)\,du = \begin{cases} 
t - X^*(t) & \text{if } 0 \leq X^*(t) \leq t, \\
0 & \text{otherwise}.
\end{cases}
\]

The case \( X^*(t) > t \) is obvious. Thus we suppose that \( 0 \leq X^*(t) < t \). Define

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for $c > 0$. The function $\alpha(c)$ is increasing. $\alpha(c)$ increases either linearly with slope 1 or by jumps. Evidently $\alpha(c-0) = \alpha(c)$ and $\alpha(c+t - \chi^*(t)) = \alpha(c)+t$ for $c > 0$. If $u = \alpha(c)$, then $\xi(u) = 1$. Since $\alpha(c) - \chi^*(\alpha(c)) = c$ and $u - \chi^*(u) \geq c$ for $u \geq \alpha(c)$, it follows that $\chi^*(u) - \chi^*(\alpha(c)) \leq u - \alpha(c)$ for $u \geq \alpha(c)$, that is, $\xi(\alpha(c)) = 1$. If $\alpha(c) < u < \alpha(c+0)$, then evidently $\xi(u) = 0$. Accordingly, in the interval $[\alpha(c), \alpha(c+t - \chi^*(t))]$ of length $t$, $\xi(u) = 1$ on the set $\{ u: u = \alpha(z) \text{ for } c \leq z \leq c+t - \chi^*(t) \}$ of measure $t - \chi^*(t)$ and $\xi(u) = 0$ elsewhere. (If $\chi^*(u)$ is defined as continuous on the right, and $z$ is a discontinuity point of $\alpha(z)$, then also $\xi(\alpha(z+0)) = 1$. However, since the discontinuity points of $\alpha(z)$ form a set of measure 0, this does not make any difference.) Since $\xi(u)$ is periodic with period $t$, (3) follows.

Now, if we suppose that $\{ \chi^*(u), 0 \leq u < \infty \}$ is the stochastic process associated with the process $\{ \chi(u), 0 \leq u \leq t \}$ under consideration and if $\xi(u)$ is defined by (2), then $\xi(u)$ is a random variable which has the same distribution for all $u \geq 0$. Thus

$$P\{ \chi(u) \leq u \text{ for } 0 \leq u \leq t \mid \chi(t) \}$$

$$= E\{ \xi(0) \mid \chi(t) \} = \frac{1}{t} \int_0^t E\{ \xi(u) \mid \chi(t) \} du$$

$$= E\left\{ \frac{1}{t} \int_0^t \xi(u) du \mid \chi(t) \right\} = \begin{cases} 1 - \frac{\chi(t)}{t} & \text{if } 0 \leq \chi(t) \leq t, \\ 0 & \text{otherwise}, \end{cases}$$

with probability 1. The last equality follows from identity (3) which now holds for almost all sample functions. This completes the proof of the theorem.