RESEARCH PROBLEMS

12. Richard Bellman: Matrix theory

All matrices that appear are $N \times N$ symmetric matrices, $N \geq 2$, and $A \geq B$ signifies that $A - B$ is nonnegative definite. Consider the set of matrices $X$ with the property that $X \geq A_1$ and $X \geq A_2$ where $A_1$ and $A_2$ are two given matrices. Let us choose an element in this set (or the element) with the property that $g(X)$, a prescribed scalar function of $X$, is minimized. For example, $g(X)$ might be $\text{tr}(X)$, $\text{tr}(X^2)$, or the largest characteristic root of $X$.

This procedure defines a function of $A_1$ and $A_2$ which we denote by $m(A_1, A_2)$. Similarly, we may define $m(A_1, A_2)$, and, generally, $m(A_1, A_2, \cdots, A_k)$ for any $k \geq 2$.

Can we find a function $g(X)$ with the property that $m(A_1, A_2, A_3) = m(A_1, m(A_2, A_3))$? If so, determine all such functions, and for general $k \geq 3$ as well. (Received April 10, 1965.)

13. Richard Bellman: Differential approximation

Let $y(t)$ be a given vector function belonging to $L^2(0, T)$ and let $x(t)$ be determined as the solution of the linear vector differential equation $x' = Ax$, $x(0) = c$. Under what conditions on $c$ and $y$ does the expression $\int_0^T (x - y, x - y) \, dt$ possess a minimum rather than an infimum with respect to the constant matrix $A$? (Received May 12, 1965.)

14. Richard Bellman: A limit theorem

It is well known that if $u_n \geq 0$ and $u_{m+n} \leq u_m + u_n$, for $m, n = 0, 1, \cdots$, then $u_n \sim nc$ as $n \to \infty$ for some constant $c$. Let $u_n(p)$ be a function of $p$ for $p \in S$, a given set, and $T(p)$ be a transformation with the property that $T(p) \in S$ whenever $p \in S$, $i = 1, 2$. Suppose that

$$u_{m+n}(p) \leq u_m T_1(p) + u_n(T_2(p))$$

for all $p \in S$ and $m, n = 0, 1, \cdots$. Under what conditions on $T_1(p)$ and $T_2(p)$ and $S$ is it true that $u_n(p) \sim ng(p)$ as $n \to \infty$? When is $g(p)$ independent of $p$? (Received May 12, 1965.)

15. Richard Bellman: Generalized existence and uniqueness theorems

Given a second-order linear differential equation $u'' + p(t)u' + q(t)u = 0$, subject to various initial and boundary conditions, there are two types of problems we can consider. The first are the classical
existence and uniqueness theorems; the second are often called "inverse problems," where the problem is that of determining the properties of the coefficients from the properties of the solution. For a version of the second type, of importance in modern physics, see, for example, B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by two of its spectra*, Russian Math. Surveys 19, No. 2 (1964), 1–64. References to earlier work by Borg, Levinson, and others will be found there.

Let us now consider a class of problems containing both of the foregoing as special cases. A simple version is the following. Suppose that $0 \leq t \leq T$, and that $S_1, S_2, S_3, S_4$ are subsets of the interval $[0, T]$. What classes of sets $S_1, S_2, S_3, S_4$, and what types of conditions on $u$ in $S_1$, $u'$ in $S_2$, $p$ in $S_3$, and $q$ in $S_4$, determine $u$, $p$, and $q$ in $[0, T]$?

Analogous problems of greater complexity and generality can be posed for higher-order and nonlinear differential equations, for partial differential equations, and for all of the classes of functional equations of analysis. In abstract form, we consider a functional equation $u = T(u, v, a)$, where $u, v$ are functions and $a$ is a vector parameter. Partial information is given concerning $u, v$, and $a$, and it is required to deduce all of the missing information. There are analogous problems for variational and control processes. (Received May 21, 1965.)

16. L. Carlitz: *A Saalschützian theorem for double series*

Saalschütz proved that

$$\text{$_pF_q$} \left[ \begin{array}{c} -m, a, b \cr c, d \end{array} \right] = \frac{(c - a)_m(c - b)_m}{(c - a - b)_m},$$

provided

$$c + d = a + b - m + 1.$$

The writer (J. London Math. Soc. 38 (1963), 415–418) proved that the series

$$S = \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r (-n)_s (a)_{r+s}(b)_r (b')_s}{r! s! (c)_{r+s} (d)_{r} (d')_s},$$

satisfies

$$S = \frac{(b + b' - a)_{m+n}(b')_{m}(b)_{n}}{(b + b')_{m+n}(b' - a)_{m}(b - a)_{n}}$$

provided
Now these conditions imply
\[*\]
\[2c + d + d' = 2a + b + b' - m - n + 2.\]
The question arises whether \( S \) can be summed when only the condition \(*\) is assumed. (Received May 15, 1965.)

17. Olga Taussky: \textit{Matrix theory}

A. It is known that the \( n \times n \) hermitian matrices are closed under the Jordan product \( AB + BA \). This composition is commutative. A noncommutative composition generalizing the Jordan product can be given for general complex matrices (or matrices over an abstract field with an involution) in the following way
\[AB + B^*A^*.\]

By \( X^* \) is meant the complex conjugate and transposed matrix. Study the structure of this generalized Jordan algebra. The idea to study this composition comes from Lyapunov’s theorem which states that a matrix whose characteristic roots have positive real parts has a unique “generalized Jordan product” hermitian inverse which is a positive definite hermitian matrix (see O. Taussky, \textit{A remark on a theorem of Lyapunov}, J. Math. Anal. Appl. 2 (1961), 105–107).

B. Let \( A \) be an \( n \times n \) matrix with complex elements and characteristic roots with negative real parts. It is known (theorem of Lyapunov) that such matrices are characterized by the fact that a positive definite matrix \( G \) exists with
\[AG + GA^* \text{ negative definite.}\]
What is the range of \( AG + GA^* \) if \( G \) runs through all positive definite \( n \times n \) matrices? \( A^* \) is the complex conjugate and transposed matrix.

C. What can one say about pairs of matrices which can be transformed to Jordan normal form simultaneously by a similarity? (Received May 20, 1965.)