A FIELD OF COHOMOLOGICAL DIMENSION 1
WHICH IS NOT $C_1$

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Communicated by A. Rosenberg, April 5, 1965

Let $k$ be a field of characteristic $q$ (= prime or 0) and let $r$ be a non-negative integer. Then $k$ is said to be $C_r$ if and only if every (homogeneous) form of degree $d$ in $n$ variables over $k$ has a nontrivial zero over $k$ if $n > d^r$. In Serre [4, Chapitre II, Corollaire to Proposition 8] the following result is obtained: If $k$ is $C_1$ then $\dim(k) \leq 1$ and $[k : k^q] = 1$ or $q$. Here $\dim(k)$ is defined cohomologically. Serre then remarks:

"On ignore si la réciproque du corollaire précédente est vrai c'est peu probable."

Actually, the problem of the relation of cohomological dimension $r$ and $C_r$ had been previously raised in Serre [3]. We exhibit below a field $R$ of characteristic zero of dimension 1 which is not $C_1$. This implies, for all $r \geq 1$, the existence of fields of dimension $r$ which are not $C_r$. But the situation is worse than that: $R$ is quasi-finite in the sense of Serre [2, Chapitre XIII, §2], and for all $r$, $R$ is not $C_r$. The interest in these considerations stemmed from a possible relation with Artin's conjecture which states: If $k$ is a totally imaginary number field or a $p$-adic field, then $k$ is $C_2$. Indeed, for such fields $k$, $\dim(k) = 2$ as is proved in Serre [4, Chapitre II, Corollaire to Proposition 12, Proposition 13].

We now define $R$. Let $F$ be an algebraically closed field of characteristic zero. If $K$ is a field, then $K((t))$ denotes the field of formal power series in $t$ over $K$. Let $F_2 = F((t_2))(t_2^{1/n} : 2 \mid n)$. If $p$ is a prime greater than 2 and $q$ is the largest prime less than $p$, we recursively define $F_p = F_q((t_p))(t_p^{1/n} : p \mid n)$. Finally we set $R = \text{inj lim}_p F_p$.

THEOREM. $R$ is quasi-finite, but $R$ is not $C_r$ for any $r$.

The proof will appear in Ax [1].

REFERENCES


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