MODELS OF SPACE-TIME

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1. Introduction. In [1] we exhibited electron spin as a nonrelativistic geometric property of (a model of) Euclidean 3-space. We now extend our model to one of space-time. The connections between 2 and 4 component spinors become lucid, while the Dirac equation and its relativistic "invariance" properties undergo a fundamental simplification and clarification.

2. Abstract space-time. We need first an axiomatic foundation strong enough to support both our mathematical considerations and their applications to physics.

DEFINITION. An $n+1$ dimensional space-time $(\mathbb{R}^{n+1})$ consists of

(A) An $n+1$ dimensional vector space $V$ over the real numbers plus a symmetric bilinear real form $A \cdot B$ (inner product) such that:

1. There exists a vector $A$ with $A \cdot A < 0$.
2. Any 2-dimensional subspace of $V$ contains a vector $A$ with $A \cdot A > 0$.

(B) A set $\chi$ of objects $p, q, \cdots$ (points or "events") plus a mapping $(p, q) \rightarrow p - q$ of $\chi \times \chi$ into $V$ such that:

1. $(p - q) + (q - r) = p - r$.
2. $p - q = 0$ implies $p = q$.
3. Given any point $q$ and any vector $A$ there exists a point $p$ with $p - q = A$.

Any $V$ satisfying (A) yields a model of space-time (vector space-time) on setting $\chi = V$. The Minkowski model $V = \mathbb{R}^{n+1}_{\mathbb{R}}$ consists of all $n+1$-tuples of real numbers $x = (x_1, \ldots, x_n, x_{n+1})$ with $x \cdot y = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$. (When $n = 3$, $x_4 = ct$, where $t$ is time and $c$ is the velocity of light.) Every $n+1$ dimensional vector space-time is isomorphic to $\mathbb{R}^{n+1}_{\mathbb{R}}$, but this result is physically misleading. Eventually we set $n = 3$, $\chi$ the physical space-time continuum, and $V = \mathbb{R}^4$, the spin model of (vector) space-time we shall construct.

3. The models $S_3$ and $W_4$. In [1] we defined the spin model $S_3$ of Euclidean 3-space as the vector space of self-adjoint linear transformations of trace 0 in a 2-dimensional unitary space $H_2$ (spin space) plus the operations $A \cdot B = (1/2)(AB + BA)$ and $A \times B = (1/2i)$

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We identify a scalar $c$ with $cI$, where $I$ is the identity transformation in $H_2$.) In general we denote the algebra of linear transformations in a vector space $E$ by $B(E)$. We summarize some results of [1] that we need:

Relative to an arbitrary orthonormal basis $\phi_1, \phi_2$ for $H_2$ any vector $A$ in $\mathbb{C}_3$ has the matrix representation

$$A \mapsto \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli matrices. Then $\mathbb{C}_3$ is 3-dimensional and

$$A \cdot A = A^2 = x_1^2 + x_2^2 + x_3^2 = -\det A.$$

Let $SU(2)$ denote the group of unitary transformations in $H_2$ of determinant 1 and $SO(3)$, the group of rotations or orthogonal transformations of determinant 1 in $\mathbb{C}_3$. Given $U$ in $SU(2)$ set $R_U A = UA U^{-1} (A \in \mathbb{C}_3).$ Then $R_U$ is a linear transformation in $\mathbb{C}_3$, and the mapping $U \mapsto R_U$ is a 2-to-1 homomorphism of $SU(2)$ onto $SO(3)$.

The obvious extension of $\mathbb{C}_3$ is the vector space $W_4$ consisting of all self-adjoint linear transformations in $H_2$. Then for any $A$ in $W_4$

$$A \mapsto \begin{pmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{pmatrix} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 + x_4$$

and $-\det A = x_1^2 + x_2^2 + x_3^2 - x_4^2 \equiv A \cdot A.$ $W_4$ is then a 3+1-dimensional vector space-time, but the corresponding inner product is hybrid:

$$A \cdot B = \frac{1}{2}(AB + BA) - \frac{1}{2}(\text{trace } B)A - \frac{1}{2}(\text{trace } A)B.$$

One can now extend the covering map above by setting $SL(2, C) = \text{the group of linear transformations in } H_2 \text{ of determinant 1}, \ E\updownarrow = \text{the homogeneous proper orthochronous Lorentz group}; \text{ i.e., the linear transformations in } W_4 \text{ that preserve the inner product, have determinant 1, and don’t exchange past and future. Given } S \text{ in } SL(2, C) \text{ set } M_S A = SAS^* (A \in W_4).$ Then $M_S$ is a linear transformation in $W_4$, and one has the extended

**Theorem 3.1.** The mapping $S \mapsto M_S$ is a 2-to-1 homomorphism of $SL(2, C)$ onto $E\updownarrow$.

This result is essentially known in matrix disguise, but the co-
ordinate-free methods of [1] afford a simpler and more incisive proof than is to be found in the literature.

Although its inner product lacks the Jordan form substituting in \( \mathbb{C}_s \), the model \( W_4 \) is appropriate to analysis of the Maxwell equations and the Weyl neutrino, as we shall show in a later paper.

4. The antiquaternion unit \( J \). What one wants is an element \( J \) in \( B(H_2) \) with real square and anticommuting with \( \mathbb{C}_s \). But the only element of \( B(H_2) \) that anticommutes with \( \mathbb{C}_s \) is 0. For the same reason no nonsingular \( U \) in \( B(H_2) \) yields the space inversion \( P: R_U A = U A U^{-1} = -A \ (A \in \mathbb{C}_s) \). We are thus led to the following

**Problem.** Find all antilinear transformations \( J \) in \( H_2 \) anticommuting with \( \mathbb{C}_s \), in particular those such that \( J^2 = \pm 1 \).

In an equivalent guise (commutativity of \( J \) with the quaternion algebra \( Q = [k U: k \geq 0, U \in SU(2)] \) (cf. [2])) we obtained in [4] the following

**Solution.** Given an arbitrary orthonormal basis \( \phi_1, \phi_2 \) in \( H_2 \), identify a vector \( x_1 \phi_1 + x_2 \phi_2 \) with the column vector

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Then every such \( J \) is of the form

\[
\begin{pmatrix}
x'_1 \\
x'_2
\end{pmatrix} = J \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \omega \begin{pmatrix}
x_2 \\
x_1
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix},
\]

whence \( J^2 = -|\omega|^2 \neq 1 \) and \( J^2 = -1 \) iff \( |\omega| = 1 \)—i.e., iff \( J \) is anti-unitary.

The normalized \( J, J^2 = -1 \), thus obtained is unique up to a phase factor and may be identified with Wigner's nonrelativistic time-inversion operator for particles of spin \( \frac{1}{2} \), but the idea goes back to Möbius: The space inversion operator \( R_A A = A J A^{-1} = -A \ (A \in \mathbb{C}_s) \) arising is independent of the scalar \( \omega \neq 0 \), whence one can regard (1) as an anti-projective transformation in homogeneous coordinates. Set \( z = x_1/x_2, z' = x'_1/x'_2 \) to obtain

\[
z' = -\bar{z}^{-1}.
\]

Now map onto the Riemann sphere, \( z \rightarrow \xi \), and note that \( \xi' \) is antipodal to \( \xi \).

We can now rewrite the defining properties of \( \mathbb{C}_s \) as follows:

\( \mathbb{C}_s \) consists of all \( T \) in \( B(H_2) \) such that

\[
i T = T^* i, \quad J T = - T^* J,
\]
while the identity $A^*JA = (\det A)^{-1}J$ for $A$ in $B(H_2)$ translates the defining properties of $SU(2)$ into:

**SU(2) consists of all $T$ in $B(H_2)$ such that**

\[
T^*iT = i, \\
T^*JT = J.
\]

These formulae are independent of the phase factor for the normalized $J$. We now pick a distinguished $J$. This amounts to putting a complex orientation on $H_2$ (cf. [4]).

### 5. The spin model $E_4$ and the group $G_4^\dagger$. **Now let $E_4$ be $H_2$ considered as a real vector space plus the new inner product**

\[
\langle x | y \rangle_+ = \sigma_0(\langle x | y \rangle).
\]

$E_4$ is a 4-dimensional Euclidean vector space. Linear and antilinear transformations in $H_2$ are then on the same footing as linear transformations in $E_4$, betraying their origin only in commutativity or anticommutativity with the now distinguished linear transformation $i$. $S = T^*$ in $B(H_2)$ implies $S = T^*$ in $B(E_4)$, while the new and old trace and determinant of a $T$ from $B(H_2)$ are connected as follows:

\[
\text{trace}_4 T = 2\sigma_0(\text{trace}_2 T), \\
\text{det}_4 T = |\text{det}_2 T|^2.
\]

**DEFINITION.** $E_4$ consists of all linear transformations in $E_4$ satisfying (3).

Clearly $E_4$ is a subspace of $B(E_4)$ containing $E_3$ and closed under $\ast$.

**Theorem 5.1.** $E_4$ consists of all elements of $B(E_4)$ of the form

\[ T = A + aJ \ (A \in E_3, \ a \text{ real}). \]

Then $T^2 = A^2 - a^2$ and we can set $T_1 \cdot T_2 = \frac{1}{2}(T_1T_2 + T_2T_1)$ to obtain a $3+1$ dimensional model of vector space-time.

Let $K = (1 + J)/2^{1/2}$. Then $K$ is orthogonal, $K^2 = J$, and $K^8 = 1$.

**Theorem 5.2.** The mapping $\tau : A \rightarrow KAK$ is an isomorphism of $W_4$ onto $E_4$ leaving $E_3$ pointwise fixed and preserving the inner product.

Since every $T$ in $B(E_4)$ admits a unique decomposition $T = T_1 + T_2$, where $T_1$, $T_2$ are respectively linear and antilinear transformations in $H_2$, the space-time $E_4$ splits naturally into space and time.

**Definition.** $G_4^\dagger$ consists of all linear transformations $T$ in $E_4$ satisfying (4).

**Theorem 5.3.** $G_4^\dagger$ is a group containing $SU(2)$ and closed under $\ast$. 
If $T \in \mathfrak{g}_4^+$ set $L_T A = T A T^{-1}$ ($A \in \mathfrak{e}_4$). Then $L_T$ is a linear transformation in $\mathfrak{e}_4$, and

**Theorem 5.4.** The mapping $T \mapsto L_T$ is a 2-to-1 homomorphism of $\mathfrak{g}_4^+$ onto $\mathfrak{e}_4^+$.

Space-inversion $P$ and time-reversal $T$ arise as follows: $P: A \mapsto JA J^{-1}, T: A \mapsto i A i^{-1}$. Let $\mathfrak{g}$ be the group of linear transformations in $E_4$ generated by $\mathfrak{g}_4^+$, $J$, and $i$.

The connection between 2- and 4-component spinors is then contained in

**Theorem 5.5.** The mapping $\psi: S \mapsto KSK^{-1}$ is an isomorphism of $\text{SL}(2, C)$ onto $\mathfrak{g}_4^+$ leaving $\text{SU}(2)$ pointwise fixed.

**Theorem 5.6.** The following diagram is commutative:

![Diagram](image)

Note that $\det_4 KSK^{-1} = \det_4 S = |\det_2 S|^2 = 1$, while $\det J = \det K = 1$ and $\det_4 i = 1$, whence $\mathfrak{g}_4^+$ (or $\mathfrak{g}$) and $\text{SL}(2, C)$ are subgroups of $\text{SL}(4, \mathbb{R})$ whose intersection is $\text{SU}(2)$.

$\mathfrak{e}_4$ is also remarkable in that it admits an explicit coordinate-free oriented volume function $\theta(A_1, A_2, A_3, A_4) = \frac{1}{4} \text{trace}_4 (i A_1 A_2 A_3 A_4 J)$, reducing to $(1/2i) \text{trace}_2 (A_1 A_2 A_3) = (A_1 \times A_2) \cdot A_3$ when $A_4 = J$ and $A_1, A_2, A_3$ lie in $\mathfrak{g}_3$ (cf. [3]). Finally, the (Clifford) algebra generated by $\mathfrak{e}_4$ is just $B(E_4)$.

6. **The Dirac operator.** Let $(g_{ij}) = \text{diag}(1, 1, 1, -1)$. Then an ordered orthonormal basis $(e)$ for $\mathfrak{e}_4$ is characterized by the identity

$$(7) \quad e_i e_j + e_j e_i = 2 g_{ij}.$$ 

Let $E_4^*$ and $\mathfrak{e}_4^*$ be the respective complexifications of $E_4$ and $\mathfrak{e}_4$, and consider the expression $\langle A u, v \rangle$, where $A$ runs over $\mathfrak{e}_4$ and $u, v$ run over $E_4^*$. Since this expression is real linear in $A$, complex linear in $u$, and complex antilinear in $v$, there exists a unique mapping $F: E_4^* \times E_4^* \rightarrow \mathfrak{e}_4^*$ such that

$$(8) \quad \langle A u, v \rangle = A \cdot F(u, v),$$
and $F(u, v)$ is complex linear in $u$ and complex antilinear in $v$. In particular, $F(u, J u)$ lies in $\mathbb{C}_4$.

Given now any ordered o.n. basis $e_1, \ldots, e_4$ for $\mathbb{C}_4$ consider smooth functions $\psi : \mathbb{C}_4 \to E_4'$ and let

$$
(9) \quad (\partial_x^4 \psi)(x) = \lim_{h \to 0} \frac{\psi(x + he_i) - \psi(x)}{h}.
$$

**DEFINITION.** The Dirac operator $D = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3 - e_4 \partial_4$. Then $D^2 = \partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_4^2$, the d'Alembertian, while the Dirac equation takes the form

$$
(10) \quad D \psi + \kappa \psi = 0 \quad (\kappa = mc/h),
$$
and the associated charge-current vector $-F(\psi, J \psi)$ satisfies the continuity equation

$$
(11) \quad \text{div} \, F(\psi, J \psi) = 0.
$$

Finally the relativistic "invariance" properties of the Dirac equation reduce to simple properties of the Dirac operator $D$.

**THEOREM 6.1** (Passive Invariance). $\langle D \psi \mid u \rangle = \text{div} \, F(\psi, u)$ ($u \in E_4'$).

If $T \in G$, let $(T \psi)(x) = T \psi(T^{-1}x) = T \psi(T^{-1}xT)$.

**THEOREM 6.2** (Active Invariance). $D T = T D$.

Proofs of the above theorems and some related results will appear elsewhere.

**References**


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