Consider an essentially positive [2] system of linear DE's, that is, any system of the form

\[ \frac{dx_i}{dt} = \sum q_{ij}(t)x_j, \quad q_{ij}(t) > 0 \text{ if } i \neq j. \]

This maps the positive hyperoctant \( C \) of real \( (x_1, \ldots, x_n) \)-space into itself. The effect of (1) in a small increment of time \( dt \) is, using formulas of Ostrowski [3], to map \( C \) into an interior polyhedral cone whose projective diameter is given for \( P = \exp[Q(t)dt] = I + Q(t)dt + \cdots \) by

\[ \ln\left\{ \sup_{i,j,k,l}(p_{ki}p_{lj}/p_{kj}p_{li}) \right\}. \]

This is maximized asymptotically (as \( dt \downarrow 0 \)) by setting \( k=i, l=j \), so that the numerator approaches 1. Hence the projective diameter of \( e^{Q(t)dt}(C) \) is, asymptotically,

\[ \Delta = -\ln\left\{ \inf_{i,j}(q_{ij}q_{ji})d^2 \right\}, \quad q_{ij} = q_{ij}(t). \]

By a basic theorem of [1], all projective distances in \( C \) are therefore contracted by a factor at most

\[ \tanh(\Delta/4) = (1 - e^{-\Delta/2})/(1 + e^{-\Delta/2}) = 1 - \psi(t)dt, \quad \text{where } \psi(t) = 2[\inf_{i,j}q_{ij}(t)q_{ji}(t)]^{1/2}. \]

This proves the following basic result.

**Lemma.** For any essentially positive system (1) of linear DE's, all projective distances in \( C \) are contracted by an asymptotic factor at most \( 1 - \psi(t)dt \) in the time interval \((t, t+dt)\), where \( \psi(t) \) is given by (4).

Integrating with respect to \( t \), we deduce the

**Theorem.** For any essentially positive system (1) of linear DE's, let \( \theta(x(t), y(t)) \) denote the projective distance in \( C \) between two solutions of (1) which are positive on \([0, \infty)\). Then

\[ \theta(x(t), y(t)) \leq \theta(x(0), y(0)) \exp\left[ -\int_0^t \psi(s)ds \right], \quad t > 0. \]

For example, consider the interesting case \( d^2x/dt^2 = p(t)x, \quad p(t) > 0. \) Then
has only one pair of off-diagonal entries, and $\psi(t) = 2\sqrt{\langle p(t) \rangle}$. Hence we have the

**COROLLARY.** Let $x(t)$ and $y(t)$ be any two solutions of $d^2x/dt^2 = p(t)x$ with $p(t) > 0$, $x(0) > 0$, $x'(0) > 0$, $y(0) > 0$, $y'(0) > 0$. Then

$$
\left| \ln \left[ \frac{x(t)y'(t)}{x'(t)y(t)} \right] \right| \leq \left| \ln \left[ \frac{x(0)y'(0)}{x'(0)y(0)} \right] \right| \exp \left[ -2 \int_0^t (p(s)) ds \right].
$$

The preceding result is helpful for proving the convergence of all positive solutions of an essentially positive system (1) to a unique limiting asymptotic behavior as $t \to \infty$.

To treat similarly $d^2x/dt^2 = p(t)x$ with $p(t) > 0$, however, the preceding technique must be modified.

**REFERENCES**

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