It has been shown by Birkhoff [2], [3] that Hilbert's projective metric [4] may be applied to a variety of problems involving linear mappings of a function space into itself. In this note we shall point out that essentially the same metric may be applied to some nonlinear mappings which frequently arise in dynamic programming [1].

Let $X$ be some set, and let $P$ denote the set of all nonnegative real-valued functions which have domain $X$ and are not identically zero. We define an extended real-valued function $\theta$ on $P \times P$ as follows:

$$
\theta(f, g) = \log \left[ \left( \sup_{x \in X} \frac{f(x)}{g(x)} \right) \cdot \left( \sup_{x \in X} \frac{g(x)}{f(x)} \right) \right].
$$

In computing the ratios, we take $0/0$ to be 1, and $a/0$ to be $\infty$ if $a \neq 0$. It is easy to show that $\theta$ is an extended pseudo-metric on $P$. $\theta(f, g) = 0$ implies that $f = \lambda g$ for some constant $\lambda > 0$. We say that a subset $P^*$ of $P$ is "metric" if $\theta$ is an extended metric on $P^*$. That is, if for any $f, g \in P^*$, $\theta(f, g) = 0$ if and only if $f = g$.

Let $L$ be a map of $P$ into $P$. If

$$
\sup_{x \in X} \frac{Lf(x)}{Lg(x)} < \sup_{x \in X} \frac{f(x)}{g(x)}
$$

such that $0 < \theta(f, g) < \infty$ then we say $L$ is "ratio reducing on $P$.” Note that if $L$ is ratio reducing on $P$ it follows at once that $\theta(Lf, Lg) < \theta(f, g)$ for all $f, g \in P$ such that $0 < \theta(f, g) < \infty$.

Thus $L$ is a contraction mapping with respect to the pseudo-metric $\theta$. Similar definitions apply on any subset of $P$. Many linear transformations have been shown [2], [3] to be ratio reducing (or at least ratio nonincreasing). A family $\{L_\lambda\}$ ($\lambda$ ranging over some set of parameters $\Lambda$) is said to be "uniformly ratio reducing" if, given $f, g$,

$$
\sup_{x \in X} \frac{L_\lambda(f(x))}{L_\lambda(g(x))} \leq \sup_{x \in X} \frac{f(x)}{g(x)} - \delta_{f, g} \quad \text{for all } \lambda \in \Lambda,
$$

where $\delta_{f, g} > 0$ may depend on $f$ and $g$ but does not depend on $\lambda$. Note that if $\Lambda$ is a finite set then the family $\{L_\lambda\}$ is uniformly ratio reducing if each of its members is ratio reducing.

**Theorem.** If the family $\{L_\lambda: \lambda \in \Lambda\}$ is uniformly ratio reducing,
then the transformation $L^1$ defined by

$$L^1(f(x)) = \sup_{\lambda \in \Delta} L_\lambda(f(x))$$

is ratio reducing. If in addition $L_\lambda(g(x)) > \delta_0 > 0$ for each $g \in P$ and all $\lambda \in \Delta$, then the transformation $L^2$ defined by

$$L^2(f(x)) = \inf_{\lambda \in \Delta} L_\lambda(f(x))$$

is also ratio reducing.

The proof of the theorem is by straightforward computation. To illustrate the application of this theorem to dynamic programming, let us consider a class of problems referred to as “equations of type III” [1, pp. 125–129]. Suppose we are confronted with a system which may be in any one of $N+1$ states (call the states $s_0, s_1, \cdots, s_N$), and we are trying to drive the system into state $s_0$. At each stage, we begin by knowing a probability distribution $p = (p_0, p_1, \cdots, p_N)$, where $p_i$ = probability that the system is in state $s_i$. We may either observe the system (at a cost $b > 0$), or we may perform an operation $T_i$ on it which will alter the probability distribution in some way at a cost $a_i > 0$ ($i = 1, 2, \cdots, n$). Then if $f(p)$ represents the expected cost of driving the system into state $s_0$ given that it is initially “known” to be in state $s_i$ with probability $p_i$, we see that $f$ must obey the functional equation

$$f(p) = \inf \left\{ \sum_{i=1}^{N} p_i f(s_i) + b, f(T_ip) + a_i \right\}$$

where $s_i$ denotes the probability distribution which assigns probability 1 to state $s_i$.

**Theorem.** There is at most one bounded positive solution to the equation (*)

**Proof.** Let $X$ be the set of all possible distributions over the $N+1$ possible states with the exception of $(1, 0, \cdots, 0)$. This point $(s_0)$ is in the closure of $X$. Since the final operation on the system must be an observation, we see that $f(p) \geq b$. If $f$ is bounded, it immediately follows that $\lim_{p \to s_0} f(p) = b$. Let us restrict our attention to the metric subset $P^*$ of $P$ consisting of bounded $f$ such that $\lim_{p \to s_0} f(p) = b$.

$$L_0(f(p)) = \sum_{i=1}^{n} p_i f(s_i) + b,$$

$$L_i(f(p)) = f(T_ip) + a_i, \quad i = 1, 2, \cdots, n,$$
are all ratio-reducing on $P^*$. Thus by our Theorem above

$$L(f(p)) = \inf_{t=0,1,\ldots,n} L_t(f(p))$$

is ratio-reducing on $P^*$. Hence, if $f$ and $g$ are distinct elements of $P^*$, then $\theta(Lf, Lg) < \theta(f, g)$, which proves there can be at most one bounded solution to $f = Lf$.

A similar method may be applied when the system may be in any one of a continuum of states. Note that in addition to proving the uniqueness of the solution (if any) to (*), the above argument shows that if $g \in P^*$, and $\{L^n g\}$ contains a uniformly convergent subsequence, then $\{L^n g\}$ converges uniformly to the solution of (*).

**BIBLIOGRAPHY**


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