THE GALOIS THEORY OF INFINITE PURELY INSEPARABLE EXTENSIONS

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Introduction. Given a field $K$ of characteristic $p \neq 0$, denote by $\text{Der}(K)$ the set of all derivations of $K$. Then $\text{Der}(K)$ is a vector space over $K$, and a Lie subring of the ring of additive endomorphisms of $K$. Moreover, $\text{Der}(K)$ is closed under $p$th powers. A Lie ring satisfying this additional closure property is called a restricted Lie ring. Take any subfield $F$ of $K$ such that $K$ over $F$ is of exponent one, i.e., $K^p \subseteq F$. Denote by $\text{Der}(K/F)$ the set of all derivations of $K$ which vanish on $F$. Then $\text{Der}(K/F)$ is a vector subspace and restricted Lie subring of $\text{Der}(K)$. On the other hand, take a restricted Lie subring $D$ of $\text{Der}(K)$ which is also a vector subspace over $K$. Let $\Phi(D) = \{ x \in K \mid \lambda(x) = 0 \text{ for every } \lambda \in D \}$. Then $\Phi(D)$ is a subfield of $K$ such that $K$ over $\Phi(D)$ is of exponent one. This gives a one-to-one correspondence between subfields $F$ of $K$ over which $K$ is finite and of exponent one, and restricted Lie subrings of finite dimension over $K$ (cf. [1] and [2]). The purpose of this note is to extend this Galois correspondence to the infinite dimensional case. The first half of the correspondence is valid regardless of the dimension of $K$ over $F$, i.e., $\Phi(\text{Der}(K/F)) = F$ if $K^p \subseteq F$ [1, p. 183]. However, to establish the second half of the correspondence, one must put a stronger condition on the vector subspace of $\text{Der}(K)$, namely, that of $p$-convexity.

$p$-convexity. Let us fix a field $K$ of characteristic $p \neq 0$. Since we shall only consider subfields $F$ for which $K^p \subseteq F$, we should designate $K^p$ as our base field. For every $x \in K$, let $H_x$ denote the set of all $\lambda$ in $\text{Der}(K)$ such that $\lambda(x) = 0$. $H_x$ may be regarded as a "distinguished" hyperplane in $\text{Der}(K)$. We call a subspace $V$ of $\text{Der}(K)$ $p$-convex if $V = \cap (V + H_x)$, the intersection being taken over all $x \in K$.

Theorem 1. Let $V$ be a vector subspace of $\text{Der}(K)$ which is $p$-convex, and let $F = \Phi(V)$. Then $\text{Der}(K/F) = V$, which implies that every $p$-convex subspace of $\text{Der}(K)$ is automatically a restricted Lie subring of $\text{Der}(K)$. Conversely, if $F$ is a subfield of $K$ containing $K^p$, then $\text{Der}(K/F)$ is $p$-convex.

Proof. Let $\lambda \in \text{Der}(K/F)$. Take any element $x$ of $K$. If $x$ is in $F$, then $\lambda(x) = 0 = \mu(x)$ for any $\mu \in V$. Suppose that $x$ is not in $F$. Let $E_x = K^p(x)$. Then $V$ restricted to $E_x$ must be a nonzero vector subspace of $\text{Der}(E_x, K)$, the set of all derivations of $E_x$ into $K$. Denote by
$V(E_z, K)$ the restriction of $V$ to $E_z$. Since $[E_z: K^p] = p$, $D(E_z, K)$ is of dimension one over $K$ [1, p. 182], so that $V(E_z, K) = D(E_z, K)$. This shows that $\lambda = \mu$ on $E_z$ for some $\mu \in V$. Therefore, in either case, we have $\lambda = \mu + (\mu - \lambda) \in V + H_z$, which proves the first assertion. Now let $F$ be a subfield of $K$ containing $K^p$. Let $\lambda \in \cap_x (\text{Der}(K/F) + H_x)$. Then for every $x \in K$, there exists an element $\mu$ of $\text{Der}(K/F)$ such that $\lambda(x) = \mu(x)$. If $x \in F$, we have $\mu(x) = 0$, so that $\lambda \in \text{Der}(K/F)$. This proves the second assertion.

Note that any restricted Lie subring $D$ of $\text{Der}(K)$ of finite dimension over $K$ is automatically $p$-convex. This follows from the second half of Theorem 1 and the fact that $D = \text{Der}(K/\Phi(D))$. However, this is not true in general for infinite dimensional restricted Lie subrings.

**Theorem 2.** Suppose that $K$ over $F$ is infinite and purely inseparable of exponent one. Then there exists an infinite dimensional restricted Lie subring $D_0$ of $\text{Der}(K)$ which is not $p$-convex.

**Proof.** Take a $p$-basis $B$ of $K$ over $F$. Of course $B$ is infinite. For every element $x_i$ of $B$ there exists a derivation $\lambda_i$ in $\text{Der}(K/F)$ such that $\lambda_i(x_i) = 1$, while $\lambda_i(x_j) = 0$ for any other $x_j$ in $B$ [1, p. 183]. Let $D_0$ be the vector subspace of $\text{Der}(K)$ spanned by the $\lambda_i$ over $K$. Let $\mu$ be any derivation in $D_0$. Then $\mu$ vanishes on $B$, except for a finite subset $B'$ of $B$, and $\mu^p$ vanishes on $B$ except for a subset of $B'$. It follows that $\mu^p \in D_0$. In exactly the same manner one can show that $\lambda\mu - \mu\lambda$ is in $D_0$ if $\lambda$ and $\mu$ are. Thus $D_0$ is a restricted Lie subring of $\text{Der}(K)$. Clearly, $\Phi(D_0) = F$, so we must show that $\text{Der}(K/F)$ contains $D_0$ properly. But this is a trivial consequence of the fact that there exists a $\mu$ in $\text{Der}(K/F)$ such that $\mu(x_i) = 1$ for every $x_i$ in $B$ [1, p. 181]. Such a $\mu$ can not be expressed as a finite linear combination of the $x_i$.

Q.E.D.

**Remark.** After finishing this note, the author has been informed that Gerstenhaber [3] has proved that the closedness with respect to the Krull topology, together with the notion of restricted subspace, characterizes the subspace $\text{Der}(K/F)$ of $\text{Der}(K)$.

**References**


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