A NEW LOCAL PROPERTY OF EMBEDDINGS

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It is known that the possible embeddings of a topological \( n-1 \) manifold \( M^{n-1} \) in the euclidean space \( E^n \) differ in the cases \( n=3 \) and \( n>3 \) in a curious way. A topological \( n-1 \) sphere can fail to be locally flat at an arbitrary finite number of points if \( n=3 \). For \( n>3 \) this cannot happen at a set consisting of a single point [2]. It is unresolved if an \( S^{n-1} \) in \( E^n \) can fail to be locally flat at a pair of points. In this note we introduce a new notion, described in detail below, called a locally weakly flat embedding and show that if a manifold \( M^{n-1} \) in \( E^n \) is locally flat at each point except possibly at the points of a finite set \( Y \) and if \( M^{n-1} \) is locally weakly flat at each point of \( Y \), then \( M^{n-1} \) is in fact locally flat at every point. In the concluding paragraph an unsolved problem is posed.

Let \( p \in M^k \subset E^n \), or more generally \( M^k \subset M^n \). Suppose \( \epsilon > 0 \). Let \( B^*_\epsilon \) be a ball of diameter less than \( \epsilon \) whose interior contains \( p \). For \( 0 < t \leq \epsilon \) let \( B_t \) denote a ball whose interior contains \( p \) and is concentric to \( B^*_\epsilon \), i.e., regard \( B_t \) as a topological product \( S^{n-1} \times [0, t] \) with \( S^{n-1} \times [0] \) identified with \( p \). For all \( t \) such that \( \epsilon - t \) is sufficiently small we hypothesize that \( B_t \cap M \) is a \( k-1 \) sphere such that the pairs

\[(E^n, B_t \cap M \times I^{n-k+1}) \approx (E^n, S^{k-1} \times I^{n-k+1})\]

are homeomorphic. If for a sequence of positive numbers \( \epsilon_1, \epsilon_2, \ldots \) converging to zero, this condition holds, we describe the embedding by saying \( M^k \) is locally weakly flat at \( p \). If this holds for all \( p \in M^k \), \( M^k \) is locally weakly flat in \( M^n \), denoted by LWF.

A comparison with other local properties of embeddings [3] shows that \( LF \Rightarrow LU \Rightarrow LWF \Rightarrow LPU \Rightarrow LPU \).

For \( n=3, k=2 \) these implications may be reversed [4]. There are examples, for \( n=3 \), that show that at a single point, local peripheral unknottedness, or local weakly flatness does not imply local flatness [5].

For \( n=3, k=1 \), LU and LPU are entirely independent. In this paper attention is restricted to \( k=n-1 \).

Theorem. Let \( M^{n-1} \subset E^n \) be a closed \( n-1 \) manifold that is locally

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flat at each point except possibly at the points of a finite set \( Y \). Suppose that \( M^{n-1} \) is LWF at each point. Then \( M^{n-1} \) is locally flat at each point.

The proof rests on an adaptation of a theorem of M. Brown’s to what I refer to as the “Turning Lemma” for annuli. The same idea can be used to establish a “Union Lemma” for \( n-1 \) disks in \( E^n \).

**Notations.** In order to ease our descriptions we define once and for all the meaning of

1. nice \( k \)-disk in \( S^k \), denoted by \( D^k \);
2. nice \( k \)-disk in \( E^{k+1} \), denoted by \( D^k \);
3. nice \( k \)-annulus in \( S^k \), denoted by \( A^k \);
4. nice \( k \)-annulus in \( E^{k+1} \), denoted by \( A^k \).

By (1) we mean the boundary \( \partial D^k \) of \( D^k \) has a shell neighborhood in \( S^k \). By (2) we mean that \( D^k \) is the image of an equatorial plane section under some homeomorphism of a standard \( k+1 \) ball into \( E^{k+1} \). By (3) we mean each boundary component of \( \partial A^k \), the boundary of \( A^k \), has a shell neighborhood in \( S^k \). By (4) we mean \( A^k \) is the image of an equatorial plane section under some homeomorphism of a standard \( I^2 \times S^{k-1} \) into \( E^{k+1} \).

**Some recent results needed for the proof.** 1. Let \( k \) be a homeomorphic embedding of \( S^n \times [-1, 1] \) into \( S^{n+1} \), where \([a, b]\) denotes the closed real number interval \( a \leq t \leq b \). Then the closure of either complementary domain of \( h(S^n \times [0]) \) is an \((n+1)\)-cell (Theorem 5 of A proof of the generalized Schoenflies theorem, M. Brown).

2. Let \( B \) be a subset of a metric space \( X \). Suppose \( B = U_1 \cup U_2 \), where \( U_1, U_2 \) are open in \( B \) and \( U_1 \cap U_2 \neq \emptyset \). If both \( U_1, U_2 \) are collared in \( X \), then \( B \) is collared in \( X \). If \( B \) is an orientable bounded manifold of dim \( n \) in \( E^{n+1} \), and \( B \) is collared on each “side,” \( B \) is bi-collared at each point of \( B \setminus \partial B \). (Lemma 4 of Locally flat embeddings of topological manifolds, M. Brown [1]).

3. Let \( D_1 \) and \( D_2 \) be topological \( n \)-disks in \( E^{n+1} \). Suppose each of \( D_1 \) and \( D_2 \) is nice (see above under Notations). Let \( D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = S^{n-1} \). Suppose \( S^{n-1} \) lies in the interior of a nice annulus \( A \) that is a subset of \( D_1 \cup D_2 \). \(^2\) Then \( S = D_1 \cup D_2 \) is nice.

3’. Let \( D \) and \( A \) be respectively a nice \( n \)-disk, a nice \( n \)-annulus in \( E^{n+1} \). Suppose \( \overline{D} \cup \overline{A} \) is a disk. Further \( \partial \overline{D} \) lies in \( \text{Int} \overline{A} \). Then \( \overline{D} \cup \overline{A} \) is a nice disk in \( E^{n+1} \). The proofs of 3 and 3’ are so similar to that of 3’ we omit them.

\(^2\) The symbol “int” occurs in two senses. The meaning will be clear since in one case it means the bounded component of the complement of a set and in the other case it refers to the points not on the combinatorial boundary of some manifold with boundary.
THE TURNING LEMMA. Let $F$ be a homeomorphism, $F: S^{n-1} \times I^2 \to E^{n+1}$. Let $I_1$ and $I_2$ be intervals lying in the interior of $I^2$ such that $I_1 \cap I_2 = \{0\}$, an endpoint of each of them. Suppose

(i) $F|S^{n-1} \times I_1 = A_1$, $F|S^{n-1} \times I_2 = A_2$, and

(ii) $F|S^{n-1} \times \{0\} = S^{n-1}$.

Then $A_1 \cup A_2$ is nice in $E^{n+1}$.

To put it another way, whenever two $n$-annuli $A_1$ and $A_2$ are nice in $E^{n+1}$ and their common part is a component $S_{12}$ of the boundary of each of them, and if $F$ satisfies the consistency conditions (i) and (ii) above, then $A_1 \cup A_2$ is nice.

PROOF. Let $g$ be a homeomorphism of $I^2$ on $I^2$ so that $I_1 \cup J_1$ is carried onto $I_2 \cup J_2$ carrying $\{0\}$ into an inner point of $J_1$, leaving the other endpoints fixed, and also leaving the points of $S' = S \cap I^2$ fixed. Then

$$G(x, y) = F(x, g(y))$$

defines a homeomorphism of $S^{n-1} \times I^2$ onto $F(S^{n-1} \times I^2)$ and $A_1$ onto $\hat{A}_1$ (say). Then $\text{Int}(A_1 \cup A_2) = \text{Int} A_1 \cup \text{Int} \hat{A}_1$ and $\text{Int} A_2 \cap \text{Int} \hat{A}_1$ is open and non-null. Then if $B = (\text{Int} A_2) \cup (\text{Int} \hat{A}_1)$, $B$ is collared, and, in fact bi-collared. Hence $A_1 \cup A_2$ is nice in $E^{n+1}$.

PROOF OF THE THEOREM. Let $p$ be a point of $Y$ and $\varepsilon$ sufficiently small that $S(p, \varepsilon) \cap (Y \setminus p) = \emptyset$ (the empty set).

Let $B_1, B_2, \cdots$ be a sequence of balls with diameter approaching zero that are “concentric” about $p$, each of which meets $M$ nicely, as guaranteed by the condition $(E^n, \hat{B}_i \cap M \times I^2) \approx (E^n, S^{n-2} \times I^2)$. The spheres $\hat{B}_i, \hat{B}_2, \cdots$ may be taken disjoint. Let $\hat{B}_i$ and $\hat{B}_{i+1}$ determine an annulus $A_i$ on $M$. Since $\hat{B}_i \cap M$ is nice in $M$, a homeomorphism of $\hat{B}_i$ onto itself moving points an arbitrarily small amount may be defined to insure $A_i$ is an annulus. The boundary components of $A_i$ are denoted by $S_i$ and $S_{i+1}$. Let $B_i$ be decomposed by $S_i$ into two components $C_i^N$ and $C_i^S$, whose closures are closed $n-1$ disks and the notation is chosen so that $C_1^N, C_2^N, \cdots$ all lie on the same side of $E^n \setminus M$. Since $S^{n-2}$ is nicely embedded in $E^n$, it is clear that the consistency conditions required in the hypotheses of 3' above hold for $S_i$ relative to $A_i$ and $C_i^N$. Hence $A_i \cup C_i^N$ is a nice disk $F_i$. Since $S_{i+1}$ is nice relative to $C_i^N$ and $A_i$, $C_{i+1}^N \cup A_i$ is a nice $n-1$ disk $G_i$. The conditions of 3 (above) are fulfilled so that $F_i \cup G_i$ is a flat $n-1$ sphere. By passing $n-1$ planes parallel to the base of an $n$-simplex that converge to the $a$
vertex, one may slice the $n$-simplex into a sequence of nice $n$-cells $\sigma^n_i, \ldots$ with diameters approaching zero. By mapping each $\sigma^n_i$ to $F_i \cup G_i$ so that the consecutive functions agree on the common face of $\sigma^n_i$ and $\sigma^n_{i+1}$, the manifold $M$ is seen to have a collar at $p$ relative to the complementary domain determined by $C^n_i, \ldots$. A similar construction of the other side of $M$ shows that $M$ is in fact locally bi-collared at $p$.

By noting that the set $Y$ of $M$ consisting of points where $M$ fails to be locally flat is closed, it is easy to extend the above theorem to the case cardinal of $Y \leq \aleph_0$.

Added in proof. Corollary. If $S$ is an $n-1$ sphere that is locally flat except possibly at two points $p$ and $q$ and if $S$ is LWF at either $p$ or $q$, then $S$ is flat.\footnote{One may define a concept of $M^{n-1}$ being LWF with respect to the complementary domain $A$ (or the other complementary domain $B$) and derive a similar result.}

A question we have been unable to resolve is contained in the following.

Problem. If $M^{n-1}$ is LWF, is it LF in $M^n$? The result is known to be true for $n = 3$.

References


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