A GENERALIZATION OF A THEOREM OF NEHARI

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In 1956 Nehari, \([N.1]\) showed that the singular points \(t_0\) of the Legendre series,

\[
\phi(t) = \sum_{n=0}^{\infty} a_n P_n(t), \quad |t + 1| + |t - 1| < \rho + \frac{1}{\rho},
\]

where \(\limsup_{n \to \infty} |a_n|^{1/n} = \rho^{-1} < 1\), are related to the singular points \(\xi_0\) of the associated power series,

\[
(2) \quad f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \quad |\xi| < \rho^{-1},
\]

by the formula \(t_0 = \frac{1}{2}(\xi_0 + 1/\xi_0)\), providing \(t_0 \neq \pm 1\). The purpose of this note is to announce similar results concerning the singularities of functions \(\phi(z)\) defined by series of the form \(\sum_{n=0}^{\infty} a_n v_n(z)\), where the \(v_n(z)\) are normalized eigenfunctions of the Sturm-Liouville system

\[
(3) \quad v''(z) + (\rho^2 - q(z))v(z) = 0, \quad v'(0) + hv(0) = v'(\pi) + Hv(\pi) = 0.
\]

Indeed we are able to establish the following result.

**THEOREM.** Let \(q(z) \in C^\infty[0, \pi]\), let the \(v_n(z)\) be the set of normalized eigenfunctions of the Sturm-Liouville system (3), and let \(\{a_n\}\) be a sequence of complex numbers such that \(\limsup_{n \to \infty} |a_n|^{1/n} = \rho^{-1} < 1\). Furthermore, let us introduce the pair of analytic function elements defined in a neighborhood of the origin,

\[
(A) \quad f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n, \quad |\xi| < \rho^{-1};
\]

\[
(B) \quad \psi(t) = \sum_{n=0}^{\infty} a_n u_n(t), \quad |t + 1| + |t - 1| < \rho + \frac{1}{\rho},
\]

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where \( u_n(\cos z) = v_n(z) \). Then, providing \( t \neq \pm 1 \), the function \( \psi(t) \), defined above is singular at those points \( t = \frac{1}{2}(\alpha + 1/\alpha) \), where \( \xi = \alpha \) is a singular point of the series \( f(\xi) \).

**Sketch of Proof.** The method of proof parallels that given originally by Nehari \([N.1]\) which has been extended by Gilbert \([G.1-4]\), and Gilbert and Howard \([G.H.1-2]\), who have used these ideas in conjunction with Bergman's integral operator method \([B.1-2]\) for the study of partial differential equations. (See also the survey paper by Gilbert, Howard and Aks \([G.H.A.1]\).)

We are able to introduce an integral operator \( \mathcal{F}[f] \), which maps power series (A) onto the eigenfunction series (B). Furthermore, we also are able to construct an inverse operator \( \mathcal{F}^{-1}[\psi] \), which maps \( \psi(t) \) onto \( f(\xi) \). These operators may be seen to have the form given below,

\[
\mathcal{F}[f] = \int_{|t|=\rho_0} K(t, \xi)f(\xi) \frac{d\xi}{\xi},
\]

where \( 1 < \rho_0 < \rho \), and

\[
\mathcal{F}^{-1}[\psi] = \int_{-1}^{+1} K(t, \xi^{-1})\psi(t) \, dt,
\]

where the integration is along the real axis, and where \( K(t, \xi) = \sum_{n=0}^{\infty} u_n(t)\xi^{-n} \). By the use of the elementary Hartogs' theorem \([B.M.1]\), and known appraisals for the \( u_n(s) \) (and hence for the \( u_n(t) \)) as \( n \to \infty \) we are able to establish that \( K(t, \xi^{-1}) \) is a holomorphic function of two complex variables in certain product domains; consequently, we may consider the above integrals as Cauchy integrals.

We are also able to determine the first analytic set (moving outwards from the origin in the \( \xi \)-plane) on which \( K(t, \xi^{-1}) \) is singular, as given by \( \{\xi^2 - 2\xi t + 1 = 0\} \). Using this information plus the argument used by Hadamard \([H.1], [N.1]\) in his proof of the "multiplication of singularities" theorem, allows us to establish the fact that if \( f(\xi) \) is singular at \( \xi = \alpha \), \( (|\alpha| = 1/\rho) \), then in the compact ellipse, \( |t+1| + |t-1| \leq \rho + 1/\rho \), \( \psi(t) \) is regular for all points \( t \neq \frac{1}{2}(\alpha + 1/\alpha) \). Correspondingly, we are able to show if \( \psi(t) \) has a singularity on the boundary of the above ellipse, say at \( t = \sigma \), \( (\sigma = \frac{1}{2}(\alpha + 1/\alpha)) \), then \( f(\xi) \) is regular at all points \( \xi \neq \alpha \), such that \( |\xi| \leq |\alpha| \). Combining these results yields the above theorem.

**Remark.** We have also been able to find analogous results for eigenfunction expansions associated with quite general \( n \)th order Sturm-Liouville systems. These results along with the details of the above proof will be published elsewhere.
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REFERENCES


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