UNIFORM ASYMPTOTIC EXPANSIONS OF THE MODIFIED BESSEL FUNCTION OF THE THIRD KIND OF LARGE IMAGINARY ORDER

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The modified Bessel equation

\[ \frac{d^2w}{dz^2} + \frac{1}{z} \frac{dw}{dz} - \left( 1 - \frac{\nu^2}{z^2} \right) w = 0 \]

with its particular solution \( w = K_\nu(z) \), the modified Bessel function of the third kind with pure imaginary order is of fundamental significance in the diffraction theory of pulses. Moreover this function is the kernel of the Lebedev transform [3].

With the exception of Friedlander's results [2] little information is available about the behavior of \( K_\nu(z) \) when both \( \nu \) and \( z \) are large which case is of great importance in the applications. In [2] Langer's differential equation method is applied and an asymptotic formula is given for the function and for the zeros of its derivative.

The aim of this paper is to give a fairly complete description of \( K_\nu(z) \) for \( \nu \to \infty \); the proofs will be given elsewhere.

Based on (1) and on Olver's Theorem B [4], [6] a uniform asymptotic series is constructed in terms of the Airy function \( Ai(\xi) \) and of its derivative \( Ai'(\xi) \) for \( \nu \to \infty \) in a region \( R \) which contains the sector \( \Re s \geq 0, \Im s > 0 \).

\[
K_\nu(\nu z) = \frac{\pi^{1/2}}{\nu^{1/2}} \exp \left( -\frac{\pi}{2} \nu \right) \left( \frac{\xi}{\nu^2 - 1} \right)^{1/4} \left\{ Ai(\xi) \left[ 1 + \sum_{s=1}^m \frac{A_s(\xi)}{\nu^{2s}} \right] \right. \\
+ \left. \frac{Ai'(\xi)}{\nu^{4/3}} \sum_{s=0}^{m-1} \frac{B_s(\xi)}{\nu^{2s}} + \exp \left\{ -3 \xi^{3/2} \right\} \frac{1}{1 + \left| \xi \right|^{1/4}} O(\nu^{-2m-1}) \right\},
\]

where \( \xi = \nu^{2/3} \zeta, \frac{2}{3} \xi^{3/2} = (z^2 - 1)^{1/2} - \text{arcsec} \ z, \) and the coefficients are given by

\[
A_s(\xi) = \sum_{m=0}^{2s} (-1)^m b_m \xi^{-3m/2} U_{2s-2m}, \quad \xi^{1/2} B_s(\xi) = \sum_{m=0}^{2s+1} (-1)^m a_m \xi^{-3m/2} U_{2s-2m+1},
\]

with

\[
1 \text{ The material of this note is contained in the author's doctoral dissertation at Oregon State University. The author wishes to thank Professor F. Oberhettinger for the suggestion of the problem and for his helpful criticism.}
\]
\[ a_0 = 1, \quad a_s = \frac{(2s + 1)(2s + 3) \cdots (6s - 1)}{s!(144)^s}, \]
\[ b_0 = 1, \quad b_s = -\frac{6s + 1}{6s - 1} a_s, \]

and

\begin{align*}
U_0 &= 1, \\
U_1 &= \frac{1}{8} \left( \frac{5}{3} v^2 + 1 \right), \\
U_2 &= v^2 \left[ \frac{9}{128} + \frac{77}{192} v^2 + \frac{385}{1152} v^4 \right], \\
U_3 &= v^3 \left[ \frac{75}{1024} + \frac{4563}{5120} v^2 + \frac{17017}{9216} v^4 + \frac{85085}{82944} v^6 \right], \quad \cdots,
\end{align*}

where \( v = 1/(z^2 - 1)^{1/2} \).

As a consequence of the uniformity of (2) we obtain the Hankel [1, p. 23] and the Debye-type series for \( z = \nu/p \) with \( p > 1 \) and \( p < 1 \), respectively (which can be derived by the method of steepest descent) by substituting the appropriate asymptotic expansions of the Airy function [5, (A6) and (A7)] in (2). If \( p \sim 1 \), the Debye-type series fail to give satisfactory approximation; for \( |z-\nu| = O(z^{1/3}) \), \( \text{Re} \, z > 0 \) an asymptotic expansion can be given in terms of the Airy function using Schoebe's method [7], this series also can be deduced from (2) as a special case.

The uniform asymptotic expansion of

\[ K'_{iv}(vz) \approx \frac{d}{d(vz)} K_{iv}(vz) \]

is derived using Olver's Theorem B. We have, for \( z \in \mathbb{R} \) and \( \nu \to \infty \),

\[ K'_{iv}(vz) \sim \frac{2^{1/3} \pi e^{-\pi v^{1/3}}}{z} \left( \frac{\xi}{z^2 - 1} \right)^{-1/4} \]

\[ \cdot \left\{ A_i(\xi) \sum_{n=0}^\infty \frac{C_n(\xi)}{\nu^{2n}} + A'_i(\xi) \sum_{n=0}^\infty \frac{D_n(\xi)}{\nu^{2n}} \right\}, \]

where

\[ D_0 = 1, \quad C_0 = \xi^{1/2} V_1 + \frac{7}{48} \xi^{-1}, \]
\[ D_1 = \frac{455}{4608} \xi^{-3} - \frac{5}{48} \xi^{-2} C_0 + V_2, \quad \cdots, \]
and

\[ V_1 = - \left[ \frac{3}{8} + \frac{7}{24} v^2 \right] v, \quad V_2 = \left[ \frac{15}{128} + \frac{33}{64} v^2 + \frac{455}{1152} v^4 \right] v^2, \ldots \]

Again this series gives the Debye-type series of \( K'_{iv}(vz) \) as special cases.

We note that the first terms in (2) and in (3) for real values of \( z \) give the corresponding asymptotic formulae derived in [2].

Since \( Ai(z) \) and \( Ai'(z) \) have only real negative zeros it can be deduced from (2) and (3) that all zeros of \( K_{iv}(vz) \) and of \( K'_{iv}(vz) \) lie asymptotically on the segment \(-1 < z < 1\) of the real axis.

By inverting (2) and (3) asymptotic expansions for the 5th zero \( k_{r,s} \) of \( K_{iv}(z) \) and for the 5th zero \( k'_{r,s} \) of \( K'_{iv}(z) \) are obtained.

\[(4) \quad k_{r,s} \sim \nu \sum_{r=0}^{\infty} \frac{p_r(\alpha)}{v^{2r}}, \]

where

\[(5) \quad p_0 = z, \quad p_1 = \alpha_1 z', \quad p_2 = \alpha_2 z' + \frac{1}{2} \alpha_1 z'', \ldots , \]

and

\[(6) \quad \alpha_1 = -B_0, \quad \alpha_2 = -[B_1 + \alpha_1 B_1^1 + \frac{1}{2} \alpha_1 B_0 + \frac{1}{2} \alpha_1 B_0^3], \]

with

\[ B_0 = 1, \]
\[ A_r = \frac{d}{d \xi} A_{r-1} + \xi B_r, \quad A_r \equiv \frac{d}{d \xi} A_r + \xi B_r, \]
\[ B_r = \frac{d}{d \xi} B_{r-1} + \frac{d}{d \xi} B_r, \quad B_r \equiv \frac{d}{d \xi} B_r + \frac{d}{d \xi} B_{r-1}. \]
\[ k_{r,s} \sim \nu \sum_{r=0}^{\infty} \frac{g_r(\beta)}{v^{2r}}, \]

where the coefficients \( g_r(\beta) \) are given by (5) with the symbols \( p_r \) and \( \alpha_r \) are replaced by \( g_r \) and \( \beta_r \), respectively. In place of (6) we have

\[ \beta_1 = -\xi^{-1} C_0, \quad \beta_2 = -\xi^{-1} (C_1 + \beta_1 C_1^1 + \frac{1}{2} \beta_1^2 C_0 + \frac{1}{2} \beta_1^3 C_0^3), \]
where

\[
C_r^{2m} = \frac{d}{d\xi} C_r^{2m-1} + \xi D_r^{2m-1}, \quad C_r^{2m+1} = \frac{d}{d\xi} C_r^{2m} + \xi D_r^{2m},
\]

\[
D_r^{2m} = C_r^{2m-1} + \frac{d}{d\xi} D_r^{2m-1}, \quad D_r^{2m+1} = C_r^{2m} + \frac{d}{d\xi} D_r^{2m}.
\]

These expansions are uniform with respect to the enumeration $s$ of the zeros and the error on terminating the expansions at the $r$th term is $O(\nu^{-2r-1})$ and $O(\nu^{-2r^{1/3} - 1/8})$, respectively. For numerical calculations the coefficients $p_r(\xi)$ and $g_r(\xi)$ may be pretabulated.

**REFERENCES**