The purpose of this paper is to introduce certain interpolation methods (interpolators) which lead to a proof of Marcinkiewicz's theorem. We start with some definitions.

An interpolation pair is a couple of Banach spaces continuously contained in a Hausdorff topological vector space $V$.

On the vector spaces $A_1 + A_2 = \{ u \in V : u = v + w, \ v \in A_1, \ w \in A_2 \}$ and $A_1 \cap A_2$ we introduce the norms

$$\|u\|_{A_1 + A_2} = \inf \{ \|v\|_{A_1} + \|w\|_{A_2} : v + w = u, \ v \in A_1, \ w \in A_2 \},$$

$$\|u\|_{A_1 \cap A_2} = \max \{ \|u\|_{A_1}, \|u\|_{A_2} \};$$

with these norms, $A_1 + A_2$ and $A_1 \cap A_2$ become Banach spaces.

An interpolator $F$ is a function defined on interpolation pairs whose values are Banach spaces $F(A_1, A_2)$ such that:

1. $A_1 \cap A_2 \subset F(A_1, A_2) \subset A_1 + A_2$, the inclusions being continuous;
2. if $(X_1, X_2)$, $(Y_1, Y_2)$ are interpolation pairs, and $T$ is a linear map of $X_1 + X_2$ into $Y_1 + Y_2$ which maps $X_1$ into $Y_1$ and $X_2$ into $Y_2$ and which decreases the norms, then $T$ is also a norm decreasing map of $F(X_1, X_2)$ into $F(Y_1, Y_2)$.

We will say that $F(A_1, A_2)$ is an intermediate space between $A_1$ and $A_2$.

The functions considered in the following are complex-valued functions defined on a totally $\sigma$-finite measure space $(M, m)$. The distribution function of $f$ is
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\[ D(f, \lambda) = m(\{x \in M: |f(x)| > \lambda\}); \]

the nonincreasing rearrangement of \( f \) is defined by

\[ f^*(t) = \inf \{\lambda: D(f, \lambda) \leq t\}, \]

and the average function of \( f \) by

\[ f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds. \]

We define

\[ (i) \frac{q}{p} \int_0^\infty f^{**}(t)^{q/p} \, dt \]

\[ \sup_{t>0} t^{1/q} f^{**}(t) \]

\[ \sup_{t>0} f^{**}(t) = \int_0^\infty f^*(t) \, dt \]

\[ \sup_{t>0} f^{**}(t) \]

if \( 1 < p < \infty, 1 \leq q < \infty \),

if \( 1 \leq p < \infty, q = \infty \),

if \( p = 1, 1 \leq q \leq \infty \),

if \( p = \infty, 1 \leq q \leq \infty \).

For \( 1 < p \leq \infty \), the set \( \{f \in L^1 + L^\infty: \|f\|_{L_{pq}} < \infty\} \) is the well-known Lorentz space \( L_{pq} \) (see, for instance, [3]) with a slightly different norm. With our definition, the \( L_{pq} \) norm of a function \( f \in L^1 \cap L^\infty \) is a continuous function of \((p, q)\) for \( 1 \leq p \leq \infty, 1 \leq q \leq \infty \), and reduces to the \( L^1 \) norm for \( p = 1 \). A direct computation shows that \( \|f\|_{L_{pq}} = \|f\|_{L^p} \) if \( f \) is the characteristic function of a measurable subset of \( M \). Another advantage of our definition is that the \( L^p \) and \( L^\infty \) spaces can be characterized as the minimum and maximum among the intermediate spaces \( C \) between \( L^1 \) and \( L^\infty \) such that \( \|f\|_C = \|f\|_{L^p} \) for every characteristic function \( f \). More precisely:

**Theorem 1.** Let \( 1 \leq p < \infty \). Let \( C \) be an intermediate space between \( L^1 \) and \( L^\infty \) such that \( \|f\|_C = \|f\|_{L^p} \) for every characteristic function \( f \). Then \( L_{p1} \subset C \subset L_{p\infty} \) continuously, and

\[ \|f\|_{L_{pq}} \leq \|f\|_C \leq \|f\|_{L_{pq}}, \]

the first inequality being valid for every \( f \in C \) and the second for every \( f \in L_{p1} \).

If \( t \) is a positive real number, we denote by \( tA \) the Banach space obtained from \( A \) by changing its norm \( \|f\|_A \) by

\[ \|f\|_tA = 1/t \|f\|_A. \]

The following theorem has been proved in [2].
Theorem 2. If \( f \in L^1 + L^\infty \) and \( t > 0 \), then \( f**(t) = \|f\|_{L^1 + L^\infty} \).

This result leads to a natural generalization of the average function.

Definition. If \( (A_1, A_2) \) is an interpolation pair and \( u \in A_1 + A_2 \), we define the average function of \( u \) with respect to \( (A_1, A_2) \) as the function

\[
\uot(t) = \uot(A_1, A_2; t) = \|u\|_{A_1 + A_2}.
\]

We now list some properties of \( \uot(t) \).
1. For every \( t > 0 \), \( \uot(t) \) is finite.
2. \( \uot(t) \) is nonincreasing and continuous, \( tu\ot(t) \) is nondecreasing.
3. \( \uot(A_1, A_2; t) = (1/t)\uot(A_2, A_1; 1/t) \).
4. \( \uot(t) \) tends to zero when \( t \) tends to infinity for every \( u \) in \( A_1 + A_2 \) if and only if \( A_1 \cap A_2 \) is dense in \( A_2 \).
5. \( tu\ot(t) \) tends to zero when \( t \) tends to zero for every \( u \) in \( A_1 + A_2 \) if and only if \( A_1 \cap A_2 \) is dense in \( A_1 \).
6. \( \sup_{t>0} \uot(t) \leq \|u\|_{A_2} \), the equality sign holds for every \( u \) in \( A_2 \) if and only if the unit sphere of \( A_2 \) is closed in \( A_1 + A_2 \).
7. \( \sup_{t>0} tu\ot(t) \leq \|u\|_{A_1} \), and the equality sign holds for every \( u \) in \( A_1 \) if and only if the unit sphere of \( A_1 \) is closed in \( A_1 + A_2 \).
8. Let \( u \in A_1 + A_2 \) and denote

\[
\phi(\delta) = \inf \left\{ \|u\|_{A_1} : (1/t)\|u - u\|_{A_1} + \|u\|_{A_2} \leq \uot(t) + \delta \right\},
\]

\[
u^*(t) = \nu^*(A_1, A_2; t) = \lim_{\delta \to 0} \phi(\delta).
\]

Then \( \nu^*(t) \) is nonincreasing and right continuous, and

\[
uot(A_1, A_2; t) = \frac{1}{t} \int_0^t \nu^*(A_1, A_2; s) \, ds.
\]

We define, for \( u \in A_1 + A_2 \), \( \|u\|_{L_{pq}(A_1, A_2)} \) as in (1), but writing \( \uot(A_1, A_2; t) \) instead of \( f**(t) \). It can be shown that \( \|u\|_{L_{pq}(A_1, A_2)} \) is a norm, and denoting

\[
L_{pq}(A_1, A_2) = \{ u \in A_1 + A_2 : \|u\|_{L_{pq}(A_1, A_2)} < \infty \},
\]

the function \( (A_1, A_2) \to L_{pq}(A_1, A_2) \) turns out to be an interpolator.

We now compute the average function in some particular cases.

Theorem 3. Let \( 1 \leq p_1 < p_2 < \infty \), \( 1/\beta = 1/p_1 - 1/p_2 \); then

\[
\nuot(L_{p_1,1}, L_{p_2,1}; t)
\]

(2) \[
= \frac{1}{p_1 t} \int_0^t f^*(s)s^{1/p_1 - 1} \, ds + \frac{1}{p_2 t} \int_0^\infty f^*(s)s^{1/p_2 - 1} \, ds.
\]
For \( p_2 = \infty \), the right-hand side of (2) reduces to the first term, with \( \beta = p_1 \).

**Theorem 4.** Let \( 1 \leq q_1, q_2 \leq \infty \), \( q_1 \neq q_2 \); then
\[
H(f) \leq f^{**}(L_{q_1,\infty}, L_{q_2,\infty}; t) \leq 2H(f),
\]
where
\[
H(f) = \inf_{0 \leq \alpha \leq 1} \sup_{r > 0} \frac{f^{**}(s)}{(1 - \alpha)(1 + \alpha s^{-1/q_1} + \alpha s^{-1/q_2})}.
\]

Using these results it is not difficult to determine the result of applying an \( L_{pq} \) interpolator to couples of Lorentz spaces.

**Theorem 5.** Let \( 1 \leq p_1 < p_2 \leq \infty \), \( 1 < k < \infty \), \( 1 \leq r \leq \infty \),
\[
1/p = 1/(p_1 k) + 1/(p_2 k'), \quad 1/\beta = 1/p_1 - 1/p_2.
\]
Then \( L_{pr} \) is continuously contained in \( L_{kr}(L_{p_1,1}, L_{p_2,1}) \), and
\[
\|f\|_{L_{kr}(L_{p_1,1}, L_{p_2,1})} \leq \left( \frac{\beta k}{\beta} \right)^{1/r'} p'\|f\|_{L_{pr}}.
\]

**Theorem 6.** Let \( 1 \leq q_1, q_2 \leq \infty \), \( q_1 \neq q_2 \), \( M_1, M_2 > 0 \), \( 1 < k < \infty \), \( 1 \leq r \leq \infty \), and
\[
1/q = 1/(q_1 k) + 1/(q_2 k'), \quad 1/\gamma = 1/q_1 - 1/q_2.
\]
Then \( L_{kr}(L_{q_1,\infty}, L_{q_2,\infty}) \) is continuously contained in \( L_{qr} \), and
\[
\|f\|_{L_{qr}} \leq \left( \frac{\gamma}{q_2 k'} \right)^{1/r'} M_1^{1/k} M_2^{1/k'} \|f\|_{L_{kr}(L_{q_1,\infty}, L_{q_2,\infty})}.
\]

The following theorem is an immediate consequence of Theorems 5 and 6.

**Theorem 7.** Let \( T \) be a linear operator defined on \( L^1 \cap L^\infty \) such that
\[
\|Tf\|_{L_{q_i,\infty}} \leq M_i\|f\|_{L_{p_i,1}} \text{ for every } f \in L^1 \cap L^\infty, \quad i = 1, 2, \text{ with } 1 \leq p_1 \leq \infty, \quad 1 \leq q_i \leq \infty, \quad p_1 < p_2, \quad q_1 \neq q_2.\]
Then, if \( 1 \leq r \leq \infty \), \( 1 < k < \infty \), \( T \) can be extended to a continuous linear operator from \( L_{pr} \) into \( L_{qr} \) such that
\[
\|Tf\|_{L_{qr}} \leq M_1^{1/k} M_2^{1/k'} \frac{\gamma}{(q_2 k')} \left( \frac{\beta^{1/r'}}{q_2 k'} \right)^{1/r'} p'\|f\|_{L_{pr}}
\]
where \( \beta \) and \( \gamma \) are the quantities defined in (3) and (4).

Theorem 7, due to A. P. Calderón (1), implies Marcinkiewicz's theorem except for the case \( p_1 = q_1 = 1 \). In order to include this case
we would need an inequality similar to (5), but replacing \( \| Tf \|_{L^q} \) in the left-hand side by
\[
\left\{ \int_0^\infty (Tf)^*(t)^{q/p} \, dt \right\}^{1/q}.
\]

Such an inequality has been obtained, even with more generality (see [4]). We can also obtain it without essential modification of our methods using the fact that
\[
(g + h)^*(t) \leq g^*(t/2) + h^*(t/2).
\]

However, the interest of our proof lies in the fact that Theorem 7 has been obtained from the general theory of interpolation and we don’t yet know if an inequality similar to (6) holds in general for \( u^*(A_1, A_2; t) \).

**Bibliography**


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