1. Introduction. This paper concerns itself with certain rings of meromorphic functions on noncompact Riemann surfaces. Let \( \Omega \) denote a noncompact Riemann surface. We denote by \( A \) the collection of all mappings of \( \Omega \) into the complex plane \( \mathbb{C} \) which are analytic on \( \Omega \). Also, we denote by \( M \) the collection of all mappings of \( \Omega \) into the Riemann sphere \( \mathbb{C} \) which are meromorphic on \( \Omega \). As is well known, \( A \) is an integral domain under the operations of pointwise addition and multiplication, and \( M \) is the field of quotients of \( A \). The rings considered here are those subrings of \( M \) which contain the ring \( A \). Such subrings will be referred to as \( A \)-rings of \( M \). In particular, \( A \) is itself an \( A \)-ring of \( M \), as is the field \( M \).

The ring \( A \) has been extensively investigated in recent years, and a considerable amount of information concerning the ideal theory of this ring has been obtained. The main result here is the theorem of Helmer [3], which asserts that every finitely generated ideal of \( A \) is actually a principal ideal of \( A \). This theorem is the basis for most of the known results on the ideal theory of \( A \), as is evident from the papers of Henriksen [4], [5], Kakutani [7], and Banaschewski [2].

We announce some results pertaining to the \( A \)-rings of \( M \), the principal one of which is a characterization of these rings (Theorem 3). Thanks to this characterization, a number of theorems concerning the ideal theory of \( A \) extend to any \( A \)-ring of \( M \), as, for example, the theorem of Helmer. Inasmuch as \( A \) is itself an \( A \)-ring, our results may be considered as generalizations of the corresponding results for \( A \).

The methods involved in the proofs of these results involve a study and exploitation of the valuation theory of \( M \), which was previously considered by Alling [1]. In particular, we make considerable use of the valuation rings of \( M \) which are also \( A \)-rings of \( M \). These rings are readily identified by means of Helmer's theorem, and they may be employed to prove many of the known results on the ideal theory of \( A \). Moreover, the arguments involved in these proofs frequently apply to any \( A \)-ring of \( M \). It is also possible to classify certain \( A \)-rings by these methods, and we are able, for example, to determine the noetherian \( A \)-rings of \( M \).

Finally, we consider the extent to which a Riemann surface is determined by its \( A \)-rings. More exactly, we can show that if two \( A \)-
rings of functions meromorphic on two noncompact Riemann surfaces are isomorphic, then the isomorphism in question is induced by a conformal or anticonformal equivalence between the two surfaces. This may be considered as a generalization of a theorem of Nakai [8], who proved this for the case where the \( A \)-rings in question are just the rings of functions analytic on the two surfaces. The proof of this result again makes use of the valuation theory of \( M \), especially the characterization of the noetherian valuation rings of \( M \) as given by Iss'sa [6]. However, the proof does not depend on the theorem of Nakai, as we derive it from a more general result (Theorem 13) concerning isomorphisms between fields of meromorphic functions. In particular, we obtain the field isomorphism theorem of Iss'sa [6] without the use of the Nakai theorem.

2. Algebraic preliminaries. Let \( D \) be an integral domain and let \( K \) be its field of quotients. We shall say that a nonempty subset \( S \) of \( D \) is a multiplicative subset of \( D \) if \( 0 \notin S \) and if \( S \) is closed under multiplication (i.e., if \( x \in S \) and \( y \in S \), then \( xy \in S \)). If \( S \) is a multiplicative subset of \( D \), the subset \( \{x/y : x \in D, y \in S\} \) of the field \( K \), to be denoted by \( S^{-1}D \), is a subring of \( K \) containing \( D \) which will be termed the ring of quotients of \( D \) with respect to \( S \). Further, a subring \( B \) of \( K \) is called a ring of quotients of \( D \) if \( B = S^{-1}D \) for some multiplicative subset \( S \) of \( D \). In the special case \( S = D - P \), where \( P \neq D \) is a prime ideal of \( D \), the ring of quotients \( S^{-1}D \), denoted by \( D_P \), is called the localization of \( D \) at \( P \).

Given a ring of quotients \( S^{-1}D \) of \( D \), a number of relations hold between the ideals of \( S^{-1}D \) and the ideals of \( D \) which do not intersect the set \( S \) (cf. [9, pp. 41-49, 218-233]). These relations are employed in the proofs of our results, as are many results from valuation theory.

3. \( A \)-rings of \( M \).  

**Definition 1.** An \( A \)-ring of \( M \) is a subring of \( M \) which contains the ring \( A \).

Our study of the \( A \)-rings of \( M \) is based on the following three theorems, and especially on the third.

**Theorem 1.** Let \( B \) be an \( A \)-ring of \( M \) and let \( P \) be a prime ideal of \( B \). Then \( B_P \), the localization of \( B \) at \( P \), is a valuation ring of \( M \) which contains \( B \). Conversely, if \( R \) is a valuation ring of \( M \) which contains \( B \), then \( R = B_P \) for some prime ideal \( P \) of \( B \).

**Theorem 2.** Let \( B \) be an \( A \)-ring of \( M \). Then \( B \) is the intersection of a collection of valuation rings of \( M \).

**Theorem 3.** Let \( B \) be an \( A \)-ring of \( M \). Then \( B \) is a ring of quotients of \( A \). In fact, \( B = S^{-1}A \), where \( S = \{f \in A : 1/f \in B\} \).
Thus the $A$-rings of $M$ are exactly the ring of quotients of $A$, and the $A$-rings of $M$ which are also valuation rings of $M$ are exactly the localizations of $A$ at its prime ideals. These results may be used to advantage in studying the $A$-rings of $M$. In view of the relations between the ideals of $A$ and those of $S^{-1}A$, we obtain extensions of a number of results on the ideal theory of $A$ to the $A$-rings of $M$, such as the following.

**Theorem 4.** Let $B$ be an $A$-ring of $M$. Then every finitely generated ideal of $B$ is a principal ideal of $B$.

**Theorem 5.** Let $B$ be an $A$-ring of $M$ and let $P$ be a nonzero, proper prime ideal of $B$. Then $P$ is contained in exactly one maximal ideal of $B$.

**Theorem 6.** Let $B$ be an $A$-ring of $M$ and let $P$ be a maximal ideal of $B$. Then the collection of all primary ideals of $B$ which are contained in $P$ is totally ordered under set inclusion.

**Theorem 7.** Let $B$ be an $A$-ring of $M$ and let $P$ be a maximal ideal of $B$. Then the intersection of any collection of prime (resp. primary) ideals of $B$ contained in $P$ is again a prime (resp. primary) ideal of $B$.

Since these theorems are all known to be valid for $A$ itself [2], they may be considered as generalizations of the ideal theory of $A$. One may also obtain a number of results on the valuation rings of $M$ which contain $A$ by the use of our characterization of $A$-rings. For example, using some results [2], [5] on the prime ideals of $A$, we have the following.

**Theorem 8.** Let $R$ be a nontrivial valuation ring of $M$ which contains $A$. Then the following are equivalent: (1) $B$ is a noetherian ring. (2) $B$ is a valuation ring of rank one. (3) $B$ is a valuation ring of finite rank. (4) $B$ is a maximal subring of $M$. (5) There exists a point $a \in \Omega$ such that $B = \{ f \in M : f(a) \neq \infty \}$.

Of particular interest are those $A$-rings of $M$ consisting of all functions in $M$ having no poles on a given subset of $\Omega$.

**Definition 2.** Given $E \subset \Omega$, we define $A(E) = \{ f \in M : f(a) \neq \infty, a \in E \}$.

Evidently $A(E)$ is the collection of functions $f \in M$ which are analytic at each point of $E$, so $A(E)$ is an $A$-ring of $M$. With suitable restrictions on the set $E$, the ring $A(E)$ must satisfy some very strong conditions.

**Theorem 9.** Let $B$ be an $A$-ring of $M$, $B \neq M$. Then the following are equivalent: (1) $B = A(E)$, where $E$ is a nonempty, relatively compact subset of $\Omega$. (2) $B$ is a noetherian ring. (3) $B$ is a principal ideal ring.
Theorem 10. Let \( B \) be an \( A \)-ring of \( M \). Then \( B = A(E) \) for some subset \( E \) of \( \Omega \) if and only if \( B \) is the intersection of a decreasing sequence of \( A \)-rings which satisfy the conditions of Theorem 9.

4. Isomorphism theorems. In order to determine the possible ring isomorphisms between two \( A \)-rings on two noncompact Riemann surfaces, we make use of a recent theorem of Iss'ssa [6], which characterizes the noetherian valuation rings of the field \( M \). This result may be stated as follows.

Theorem 11. Let \( R \) be a noetherian valuation ring of \( M \). Then \( R \) is an \( A \)-ring of \( M \).

This result may be combined with Theorem 8 to yield

Theorem 12. Let \( R \) be a nontrivial noetherian valuation ring of \( M \). Then there exists a point \( a \in \Omega \) such that \( R = R_a = \{ f \in M : f(A) \neq \infty \} \). Conversely, for each \( a \in \Omega \) the ring \( R_a \) is a nontrivial noetherian valuation ring of \( M \).

With this result we then obtain the following theorem concerning field isomorphisms between fields of meromorphic functions.

Theorem 13. Let \( \Omega_1 \) and \( \Omega_2 \) be Riemann surfaces, where \( \Omega_1 \) is noncompact. Let \( F_2 \) be a subfield of the field of functions meromorphic on \( \Omega_2 \), \( F_2 \) containing the constants. Let \( M_1 \) denote the field of functions meromorphic on \( \Omega_1 \), and suppose that \( \theta : M_1 \to F_2 \) is a field isomorphism of \( M_1 \) onto \( F_2 \). Then there exists a unique map \( \phi : \Omega_2 \to \Omega_1 \) such that one of the following holds:

1. \( \phi \) is analytic and \( \theta f = f \circ \phi \) for all \( f \in M_1 \),
2. \( \phi \) is conjugate-analytic and \( \theta f = (f \circ \phi)^* \) for all \( f \in M_1 \).

Now if \( \Omega \) is a noncompact Riemann surface, and if \( B \) is an \( A \)-ring of \( M \), then \( M \) is the field of quotients of \( B \). Hence Theorem 13 may be applied to ring isomorphisms between \( A \)-rings on noncompact Riemann surfaces. It results that a noncompact Riemann surface \( \Omega \) is uniquely determined to within a conformal or an anti-conformal equivalence by the algebraic structure of any of the \( A \)-rings of \( M \).
This may be considered as a generalization of the theorem of Nakai [8], who proved this result for the case where the $\mathcal{A}$-ring in question is the ring $\mathcal{A}$ itself. It also contains the field isomorphism theorem of Iss'sa [6], the case where the $\mathcal{A}$-ring involved is simply the field $\mathcal{M}$.

**References**


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