EXTENSION OF NONLINEAR CONTRACTIONS

BY STEN OLOF SCHÖNBECK

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The following problem was suggested as a research problem by R. A. Hirschfeld in Bull. Amer. Math. Soc. 71 (1965), 495:

$E$ and $F$ are Banach spaces, $F$ reflexive, $D$ is a subset of $E$ and $T: D \to F$ a nonlinear contraction, i.e. $\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\|$ whenever $x_1, x_2 \in D$. Can $T$ be extended to a contraction $T': E \to F$?

Hirschfeld observes that the answer is "yes" when $E = F = \text{Hilbert space}$. The following simple example shows that the answer is "no" in general. In the two-dimensional plane $R^2$ consider a regular hexagon $H$, with its center at the origin, and a circle $C$ inscribed in $H$. Let $E$ and $F$ be $R^2$ equipped with norms $\| \cdot \|_E$ and $\| \cdot \|_F$ defined by $\{ x; \|x\|_E = 1 \} = H$, $\{ x; \|x\|_F = 1 \} = C$. Let $x_1$ and $x_2$ be two consecutive points of contact between $H$ and $C$. Then

$$\|x_1\|_E = \|x_2\|_F = \|x_2\|_E = \|x_1 - x_2\|_F = \|x_1 - x_2\|_E = 1$$

so that if $D = \{ 0, x_1, x_2 \}$ and $T(0) = 0, Tx_1 = x_1, Tx_2 = x_2$, $T$ is a contraction of $D$ into $F$. Now, if $z = (x_1 + x_2)/3$, it is easily seen that $\|z\|_E = \|z - x_1\|_E = \|z - x_2\|_E = 1/2$. Hence, if $T$ could be extended to a contraction $T': E \to F$, then the point $u = T'z$ would satisfy

$$\|u\|_F \leq 1/2, \quad \|u - x_1\|_F \leq 1/2, \quad \|u - x_2\|_F \leq 1/2$$

which is clearly impossible.

We have, however, been able to prove some positive results. In order to state these results, we introduce the following terminology. If $E$ and $F$ are normed linear spaces, we say that $(E, F)$ has the contraction-extension (c.e.) property if: for any subset $D \subseteq E$ and any contraction $T: D \to F$ there is an extension of $T$ to a contraction $T': E \to F$.

We then have

**Theorem 1.** If $E$ and $F$ are real or complex Banach spaces, if $F$ is strictly convex and if $(E, F)$ has the c.e. property, then $E$ and $F$ are Hilbert spaces.

**Outline of Proof.** It is clearly sufficient to assume that $E$ and $F$ are real spaces. Using the strict convexity of $F$, it is then easy to show that, if $x, y \in E$, $u, v \in F$ and if $\|x\| = \|u\|, \|y\| = \|v\|, \|x - y\| = \|u - v\|$, then $\|ax + by\| \geq \|au + bv\|$ for all real numbers $a, b$. 99
If $x$ and $y$ are elements or a real normed linear space, we say that $x$ is normal to $y$ if $\|x+ay\| \geq \|x\|$ for all real numbers $a$, and then we write $xNy$. Using our above result and a limiting process we may prove: if $x, y \in E, u, v \in F$ and if $\|x\| = \|u\|, \|y\| = \|v\|, xNy, uNv$, then $\|ax+by\| \geq \|au+bv\|$ for all $a, b$.

With the aid of this result, it is now possible to show that normality is a symmetric relation in both $E$ and $F$. Day [2] has given a construction of all two-dimensional spaces with symmetry of normality. By means of this construction and our previous results we may conclude: if $x, y \in E, u, v \in F$ and if $\|x\| = \|u\|, \|y\| = \|v\|, xNy, uNv$, then $\|ax+by\| = \|au+bv\|$ for all $a, b$. This implies that both $E$ and $F$ have the following property, formulated for a normed linear space $L$:

There is a single-valued function $f$ of two real variables so that for any $x, y \in L$ such that $xNy$ we have $\|x+y\| = f(\|x\|, \|y\|)$.

But this property is characteristic of euclidean (i.e. prehilbert) spaces, as can be shown in a number of ways. (See for instance Hopf [4], where this is shown even without assuming symmetry of the norm.)

**Theorem 2.** The following two properties of a real Banach space $F$ are equivalent:

(i) $(E, F)$ has the c.e. property for every real Banach space $E$

(ii) any family of closed spheres in $F$, such that any two members of it intersect, has a nonempty intersection.

**Outline of Proof.** (i)$\Rightarrow$(ii) is proved by first observing that, for any set $S$, the Banach space $m(S)$ of all bounded real-valued functions on $S$ with the supremum norm, has property (ii). We then embed $F$ isometrically in a suitable $m(S)$. If $(S_i), i \in I$, are closed spheres in $F$ such that $S_i \cap S_j \neq \emptyset$ for all $i, j$, then for the corresponding spheres $\sum_i$ in $m(S)$ we have $\cap_i \sum_i \neq \emptyset$. Using the c.e. property of $(m(S), F)$ we then conclude that $\cap_i S_i \neq \emptyset$.

(ii)$\Rightarrow$(i) is proved by Zorn's lemma in a straightforward way.

Theorem 2 shows the intimate connection between our present problem and the problem of linear, norm-preserving extension of continuous linear transformations. In fact, it has been proved by Nachbin [6] that property (ii) for a real Banach space $F$ is equivalent to

(iii) for any real Banach space $E$, any closed linear subspace $S$ of $E$ and any continuous linear transformation $T$ of $S$ into $F$, there exists a linear extension $T'$ of $T$ to $E$ with values in $F$ and $\|T'\| = \|T\|$.
Moreover, through the work of Aronszajn-Panitchpakdi [1], Goodner [3], Kelley [5] and Nachbin, it is also known that a real space $F$ has property (iii) if and only if $F$ is linearly isometric to a space $C(S)$, the space of real-valued continuous functions on a compact, Hausdorff and extremally disconnected space $S$. (For a survey of these and related problems, see Nachbin [7].)

Thus we have the following

**Corollary to Theorem 2.** If $F$ is a real Banach space, then $(E, F)$ has the c.e. property for every real Banach space $E$ if and only if $F$ is linearly isometric to a space $C(S)$, where $S$ is compact, Hausdorff and extremally disconnected.

Finally, using the corollary it is easy to show that a complex Banach space $F$ can never have property (ii). Hence we may conclude that there is no complex Banach space $F$ such that $(E, F)$ has the c.e. property for every complex Banach space $E$.

**References**


University of Stockholm, Sweden