ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR VOLTERRA EQUATIONS

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In this note we show how certain known results for delay differential equations can be extended to systems of integral equations of the form

\[ x(t) = f(t) + \int_0^t a(t - s)g(s, x(s)) \, ds \quad (t \geq 0). \]

We make the following assumptions:

(A1) \( f(t) \) is uniformly continuous and bounded on \( 0 \leq t < \infty \),

(A2) \( a(t) \) is a square matrix whose entries are \( L_1(0, \infty) \),

(A3) \( g(t, x) \) is continuous in \((t, x)\) for \( 0 \leq t < \infty \), \( |x| < \infty \) and \( g \) is uniformly almost periodic in \( t \) uniformly on compact subsets of \( x \) in real \( n \)-space \( \mathbb{R}^n \), and

(A4) \( x(t) \) is a bounded solution of (1) for \( 0 \leq t < \infty \).

Let \( \Omega \) be the positive limit set of \( x(t) \). We refer to [2] for the definitions and properties of almost periodic functions and limit sets. The analog for integral equations of [2, Theorem 1] is

**Theorem 1.** If (A1)–(A4) are satisfied, then to each point \( z \) in \( \Omega \) there corresponds a sequence \( t_m \to \infty \) as \( m \to \infty \) and functions \( G(t, x) \), \( X(t) \) and \( F(t) \) such that

(i) \( \lim_{m \to \infty} |x(t + t_m) - X(t)| + |f(t + t_m) - F(t)| = 0 \) uniformly on compact subsets of \( -\infty < t < \infty \),

(ii) \( \lim_{m \to \infty} g(t + t_m, x) = G(t, x) \) uniformly for all \( t \) and for \( x \) on compact sets, and

(iii) on the interval \( -\infty < t < \infty \), \( X(t) \in \Omega \) and

\[ X(t) = F(t) + \int_{-\infty}^t a(t - s)G(s, X(s)) \, ds. \]

**Proof.** As is well known in harmonic analysis the convolution of an \( L_1 \) function with an essentially bounded function yields a uniformly continuous function. Hence \( x(t) \) is bounded and uniformly continuous on the interval \( 0 \leq t < \infty \).

Given \( z \) in \( \Omega \) let \( \{t_m\} \) be a sequence such that \( t_m \to \infty \) and \( x(t_m) \to z \) as \( m \to \infty \). Define \( x_m(t) = x(t + t_m) \) and \( f_m(t) = f(t + t_m) \) for \( t \geq -t_m \). Since

\[ x_m(t) = f_m(t) + \int_{-t_m}^t a(t - s)g(s + t_m, x_m(s)) \, ds, \]

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the proof can now be completed in the same way as the proof of [2, Theorem 1].

We remark that with essentially the same proof one can establish a modified version of Theorem 1 in which the lower limit of integration in equation (1) is $-\infty$. Note also that one could add to the right side of (1) a bounded measurable function $h(t) \to 0$ as $t \to \infty$.

Since a bounded continuous function must tend to its positive limit set, Theorem 1 above can sometimes be used to obtain results on the asymptotic behavior of solutions. We shall illustrate the technique with some examples. Consider the scalar equation

$$x(t) = f(t) - \int_0^t a(t - s)x(s) \, ds.$$  

Paley and Wiener [4, pp. 58–63] prove:

**Theorem 2 (Paley-Wiener).** Suppose $a(t)$ is $L_1(0, \infty)$ and $f(t)$ is bounded, measurable and tends to a limit $f_0$ as $t \to \infty$. For each such $f$ the solution of (2.1) is bounded and tends to the limit

$$x(t) \to x_0 = f_0 \sqrt{1 + \int_0^\infty a(s) \, ds}$$

if and only if when $\text{Re}(u) \geq 0$ one has

$$\int_0^\infty a(t) \exp(-ut) \, dt \neq -1.$$  

To this we add

**Corollary 1.** Let $a(t)$ and $f(t)$ be as in Theorem 2. All bounded solutions of (3) satisfy (4) if and only if (5) holds whenever $\text{Re}(u) = 0$.

Under the hypothesis of Corollary 1 some solutions may be unbounded as $t \to \infty$. If we do have a bounded solution, then Theorem 1 above applies. The limiting system corresponding to (2) is in this case

$$X(t) = f_0 - \int_{-\infty}^t a(t - s)X(s) \, ds.$$  

The transformation $Y(t) = X(t) - x_0$ gives

$$Y(t) = -\int_{-\infty}^t a(t - s)Y(s) \, ds \quad (\text{for } -\infty < t < \infty).$$

For this last equation it is known that $Y(t) \equiv 0$ is the only bounded
solution if and only if the Fourier transform of $a(t)$ is never $-1$, cf. [4, p. 59 and p. 63].

Levin [1] has proved a nonlinear version of Theorem 2. Consider

$$x(t) = f(t) - \int_0^t a(t - s)g(x(s)) \, ds,$$

with the following assumptions:

(B1) $f$ is bounded and measurable on $0 \leq t < \infty$ and tends to $f_0$ as $t \to \infty$,

(B2) $g(x)$ is $C(0, \infty)$, $g(0) = 0$, and $g$ is strictly increasing, and

(B3) $a(t)$ is $C[0, \infty)$, $C^1(0, \infty)$ and $L_1(0, \infty)$, $a(t) \geq 0$, $a'(t) \leq 0$ and $a''(t) \not= 0$ on any interval except possibly $a''(t) \equiv 0$ for all large $t$.

It is possible to separate the boundedness criterion in Levin's problem in the same way that Corollary 1 refines Theorem 2. The limiting system for (6) is

$$X(t) = f_0 - \int_{-\infty}^t a(t - s)g(X(s)) \, ds.$$

Assumptions (B2) and (B3) insure that (7) has a unique constant solution $x_0$. Moreover, one can show that $x(t) \equiv x_0$ is the only bounded solution of (7). This proves

**Theorem 3.** If (B1)-(B3) hold and if $x(t)$ is a bounded solution of (6), then $x(t) \to x_0$ as $t \to \infty$.

From Levin's results we see that if, in addition, $f''(t)$ exists and is $L_1(0, \infty)$, then all solutions of (6) exist and are bounded for positive $t$. Other criterion can be given for boundedness. For example suppose $f(t)$ is bounded, $a(t)$ is $L_1(0, \infty)$ with $a(t) \geq 0$ almost everywhere and $g(x) = \exp(x) - 1$. If $x(t)$ is a solution of (6), then for as long as it exists

$$x(t) \geq f(t) - \int_0^t a(t - s) \, ds > -M.$$

Hence we also have

$$x(t) \leq f(t) + \int_0^t a(t - s)g(-M) \, ds < N.$$

By general results of Nohel [3], $x(t)$ exists and is bounded on the interval $0 \leq t < \infty$. 
REFERENCES


