ROLLING

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If $k$ is a tame arc in the 3-dimensional half-space $R^3_+ = (x, y, z, t: z \geq 0, t = 0)$ that spans the plane $R^2 = (x, y, z, t: z = 0, t = 0)$ then a locally flat 2-sphere $S$ in the 4-dimensional space $R^4 = (x, y, z, t)$ is generated by $k$ when $R^3_+$ is rotated about $R^2$. Nowadays the sphere $S$ is said to be derived from $k$ by spinning. By knotting $k$ in various ways, various types of knotted spheres $S$ can be obtained [1], but it is known that not every type of (locally flat) knotted sphere can be so obtained [2].

Some years ago I considered spheres $S$ that are obtained from $k$ by combining the spinning process with a simultaneous rotation of $k$ about its "axis" (in $R^3_+$). This operation has come to be known as twist-spinning. The specific question that I raised at that time—whether the sphere obtained by twist-spinning a trefoil $3_1$ (using a simple twist) is actually knotted—has been answered (in the negative) recently by C. Zeeman [3].

In this note I want to introduce another variation of the spinning process, one that I call roll-spinning. It is the same as twist-spinning except that instead of twisting, i.e. rotating $k$ about its axis in $R^3_+$, I roll the knot along its axis. This operation (whose name derives from its resemblance to the operation of "rolling a stocking") is somewhat difficult to describe in totally precise terms, and I will content myself here with referring to Figure II, in [3], where it is shown how to roll a figure-eight knot $4_1$.

My objective is to show that roll-spinning is not just twist-spinning in disguise (a state of affairs that one might suspect to be so). Specifically I shall show that a simple roll-spin of $4_1$ produces a type of knotted sphere $S$ that cannot be obtained from $4_1$ by any twist-spin.

Figure I gives a projection of $4_1$ with the meridian elements of its group $G$ indicated by $x, a, b, c$. From this figure the presentation

$$(x, a, b, c: ab = bx, cb = ac, cx = xb)$$

is read off in the usual way [2], [4]. If we give $4_1$ the twist-spin in which $4_1$ is rotated about its axis $n$ times the group $\Gamma_n$ of the resulting sphere (cf. [2], [3]) has presentation

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which simplifies to

\[(x, c: xcx^{-1}cx = cx^{-1}xc, x^n c = cx^n)\]

The first elementary ideal (cf. [4]) is therefore

\[\gamma_n = (1 - 3t + t^2, 1 - t^n)\].

Figures I . . . V give a sequence of projections of $4_1$ that show the successive stages of a simple "rolling." From this we obtain by the same method [2], [3] the following presentation of the group $G_1$ of the resulting knotted sphere:

\[(x, a, b, c: ab = bx, cb = ac, cx = xb, x^n c = cx^n)\]

\[b^{-1}cb = a, b^{-1}cx^{-1}b = b, b^{-1}ca^{-1}b = c\]
which simplifies to

\[(x, c: cc = x, x^2 = c^3)\].

In this case the first elementary ideal is

\[g_1 = (1 - t + t^2, 2)\].

To show that \(g_1 \neq \gamma_n\), we have only to map the ring of \(L\)-polynomials in \(t\) into the ring of integers of the cyclotomic field \(K(\omega)\), where \(\omega = e^{2\pi i/3}\), by \(t \mapsto \omega\), and note that

\[g_1 \mapsto (2),\]

\[\gamma_n \mapsto (4) \quad \text{if } n \equiv 0 \pmod{3}\]
\[\quad \mapsto (2, 1 - \omega) \quad \text{if } n \equiv 1 \pmod{3}\]
\[\quad \mapsto (2, 1 - \omega^2) \quad \text{if } n \equiv 2 \pmod{3}.\]

To put the above considerations into proper perspective one should consider the space whose elements are the arcs \(k\) in \(R^2\) that span \(R^2\). Then twist- and roll-spinning appear as loops in this "configuration space." Thus we are led to consider relative (2-dimensional) braid groups. From the methodological point of view initiated in [5], this concept generalizes the ordinary (i.e. absolute 1-dimensional) braid groups. Absolute 2-dimensional braid groups were considered in [6], and the relative 2-dimensional braid groups that I have indicated here will be the subject of a more systematic study that C. H. Giffen is undertaking.

BIBLIOGRAPHY


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