MULTILINEAR LEBESGUE-BOCHNER-STIELTJES INTEGRAL

BY WITOLD M. BOGDANOWICZ

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In this paper we introduce an integral of the form \( \int u(f_{ji}, d\mu_j) \) where 
\( u \) is a multilinear operator from the product of the Banach spaces 
\( Y_{ji}, Z_j \) \( (j=1, \ldots, m, i=1, \ldots, k_j) \) into a Banach space \( W \), and 
\( f_{ji} \) are Lebesgue-Bochner summable functions, and \( \mu_j \) are vector volumes.

The above integral is a generalization of the integral \( \int u(f, d\mu) \) 
developed in [1]. An integral similar to the last integral, developed 
in a different way, one can find in Bourbaki [10, Chapter V, p. 48–49]. For applications, see the following paper in this volume.

1. Properties of vector volumes. Let \( R \) be the space of reals and 
\( Y_i, Z_i, W \) be seminormed spaces. Denote by \( L(Y_1, \ldots, Y_k; W) \) the 
space of all \( k \)-linear continuous operators \( u \) from the space \( Y_1 \times \cdots \times Y_k \) into the space \( W \). The norms of elements in the above spaces 
will be denoted by \( |\cdot| \).

The family of sets \( V \) of an abstract space \( X \) will be called a prering 
if for any two sets \( A_1, A_2 \in V \) we have \( A_1 \cap A_2 \in V \) and there exists 
disjoint sets \( B_1, \ldots, B_k \in V \) such that \( A_1 \setminus A_2 = B_1 \cup \cdots \cup B_k \).

A nonnegative function \( v \) on a prering \( V \) is called a positive volume 
or when there is no confusion just volume if it is countably additive, 
that is for every countable family of disjoint sets \( A_t \subset V (t \in T) \) such that 
\( A = \bigcup_T A_t \in V \) we have \( v(A) = \sum_T v(A_t) \).

A function \( \mu \) from a prering \( V \) into a Banach space \( Z \) is called a 
vector volume or simply volume when there is no confusion possible 
if the function \( \mu \) is finite additive on \( V \) and for some positive volume \( v \) 
we have 
\[ |\mu(A)| \leq v(A) \] 
for all \( A \in V \).

It follows from this definition and from the definition of a prering 
that every volume is countably additive.

Theorem 1. Let \( V_i \) be a prering of sets of a space \( X_i \) \( (i=1, \ldots, k) \). 
Denote by \( V = V_1 \times \cdots \times V_k \) the family of all sets of the form \( A = A_1 \times \cdots \times A_k \) where \( A_i \in V_i \). Then \( V \) is a prering of sets of the 
space \( X = X_1 \times \cdots \times X_k \).

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A triple \((X, V, v)\), where \(V\) is a prering of sets of the space \(X\) and \(v\) is a positive volume on the prering \(V\) will be called a volume space.

**Theorem 2.** Let \((X_i, V_i, v_i)\) \((i = 1, \ldots, k)\) be volume spaces. Then the triple \((X, V, v)\), where \(X = X_1 \times \cdots \times X_k\), \(V = V_1 \times \cdots \times V_k\), and \(v(A) = v_1(A_1) \cdots v_k(A_k)\) for \(A = A_1 \times \cdots \times A_k \in V\), is a volume space. The triple \((X, V, v)\) will be called the product of the volume spaces \((X_i, V_i, v_i)\).

**Theorem 3.** Let \(V_i\) be a prering of sets of a space \(X_i\) \((i = 1, \ldots, k)\). Let \(v\) be a positive volume on \(V = V_1 \times \cdots \times V_k\) and let \(u(A_1, A_2, \ldots, A_k)\) be a function from the prering \(V\) into a Banach space \(Z\) finite additive with respect to every variable \(A_i\) separately. Then if

\[
|u(A_1, \ldots, A_k)| \leq v(A_1 \times \cdots \times A_k) \quad \text{for all } A_1 \times \cdots \times A_k \in V,
\]

the function \(u\) defined by the formula \(\mu(A_1 \times \cdots \times A_k) = u(A_1, \ldots, A_k)\) is a vector volume on the prering \(V\).

Let \((X, V, v)\) be a fixed volume space. Denote by \(M(v, Z)\) the set of all volumes \(\mu\) from the prering \(V\) into the Banach space \(Z\) such that

\[
|\mu(A)| \leq cv(A) \quad \text{for all } A \in V.
\]

The smallest constant satisfying the last inequality will be denoted by \(\|\mu\|\). It is easy to see that the space \((M(v, Z), \|\|)\) is a Banach space.

**Theorem 4.** Let \((X, V, v)\) be the product volume space of the volume spaces \((X_i, V_i, v_i)\) \((i = 1, \ldots, k)\). If \(\mu_i \in M(v_i, Z_i)\) for \(i = 1, \ldots, k\) and \(u \in L(Z_1, \ldots, Z_k; W)\) then \(\mu \in M(v, W)\) and \(\|\mu\| \leq \|u\|\|\mu_1\| \cdots \|\mu_k\|\) where \(\mu(A_1 \times \cdots \times A_k) = u(\mu_1(A_1), \ldots, \mu_k(A_k))\) for all \(A \in V\).

The proof of the theorem follows immediately from the previous one.

2. Multilinear integrals and some relations between them.

**Lemma 1.** Let \((Y_i, |\cdot|_i)\) be a family of seminormed spaces and let \(E_i\) be a dense subspace of the space \(Y_i\) \((i = 1, \ldots, k)\). If \(u\) is a \(k\)-linear operator from \(E_1 \times \cdots \times E_k\) into a Banach space \(W\) and

\[
|u(y_1, \ldots, y_k)| \leq |u| |y_1|_1 \cdots |y_k|_k
\]

for \(y_i \in E_i\) \((i = 1, \ldots, k)\) then the operator \(u\) has a unique extension to a \(k\)-linear operator \(u'\) such that \(|u'(y_1, \ldots, y_k)| \leq |u| |y_1|_1 \cdots |y_k|_k\) for \(y_i \in Y_i\) \((i = 1, \ldots, k)\).

Denote by \(S(Y)\) the family of all functions of the form
\[ h = y_1 A_1 + \cdots + y_k A_k, \] where \( A_i \in V \) is a finite family of disjoint sets and \( y_i \in Y \).

In [1] was developed the theory of the space \( L(v, Y) \) of Lebesgue-Bochner summable functions \( f \) generated by a volume space \((X, V, v)\) with values in a Banach space \( Y \). The set \( S(Y) \) according to Lemma 1 and Lemma 4, [1] is linear and dense in the space \( L(v, Y) \).

Let
\[ (X_{ji}, V_{ji}, v_{ji}) \quad (j = 1, \ldots, m; i = 1, \ldots, k_j) \]
be a family of volume spaces and let \((X_j, V_j, v_j)\) be the product of the above volume spaces corresponding to a fixed \( j \).

Let \( u \) be a multilinear continuous operator from the product of the Banach spaces \( Y_{ji}, Z_j \) \((j = 1, \ldots, m; i = 1, \ldots, k_j)\) into a Banach space \( W \).

Let \( \mu_j \in M(v_j, Z_j) \) and \( s_{ji} \in S(Y_{ji}) \). Take a representation
\[ s_{ji} = \sum_{n_{ji}} y_{n_{ji}} A_{n_{ji}}, \]
where
\[ y_{n_{ji}} \in Y_{ji} \quad \text{and} \quad A_{n_{ji}} \in V_{ji} \]
are disjoint sets. Define
\[ \int s_{ji}, d\mu_j = \sum_j \sum_i \sum_{n_{ji}} u(y_{n_{ji}}, \mu_j(A_{n_{ji}} \times \cdots \times A_{n_{jk_j}})). \]

It is easy to see that the above operator is well defined, from the product of the spaces \( U, S(Y_{ji}), M(v_j, Z_j) \) \((j = 1, \ldots, m; i = 1, \ldots, k_j)\) into the space \( W \) and
\[ \left| \int s_{ji}, d\mu_j \right| \leq |u| \left( \prod_j \|s_{ji}\| \right) \prod_j \|\mu_j\| \]
for all \( u \in U, s_{ji} \in S(Y_{ji}), \mu_j \in M(v_j, Z_j) \).

Using Lemma 1 we can extend the above operator to an operator \( \int u(f_{ji}, d\mu_j) \) defined on the product of the spaces \( U, L(v_{ji}, Y_{ji}), M(v_j, Z_j) \).

Thus we have the following

**Theorem 5.** The operator \( \int u(f_{ji}, d\mu_j) \) is multilinear from the product of the spaces \( U, L(v_{ji}, Y_{ji}), M(v_j, Z_j) \) \((j = 1, \ldots, m; i = 1, \ldots, k_j)\) into the space \( W \) and
\[ \left| \int u(f_{ji}, d\mu_j) \right| \leq |u| \left( \prod_j \|f_{ji}\| \right) \left( \prod_j \|\mu_j\| \right) \]
for all \( u \in U, f_i \in L(v_{ji}, Y_{ji}), \mu_j \in M(v_{ji}, Z_j) \).

**Theorem 6.** Let \( (X, V, v) \) be the product of volume spaces \( (X_j, V_j, v_j) \) \((j = 1, \cdots, k)\) and let \( f_i \in L(v_{ji}, Y_{ji}) \). Let \( u \) be a \( k \)-linear continuous operator from the space \( Y_1 \times \cdots \times Y_k \) into \( W \). Then the function \( f \) defined by the formula

\[
  f(x_1, \cdots, x_k) = u(f_1(x_1), \cdots, f_k(x_k))
\]

on the space \( X \) belongs to the space \( L(v, W) \) and

\[
  \|f\| \leq |u| \|f_1\| \cdots \|f_k\|.
\]

Let \( (X, V, v) \) be the product of the volume spaces \( (X_j, V_j, v_j) \) where \( j = 1, \cdots, k \).

Let \( Y_i, Z \) be Banach spaces. Consider a multilinear operator \( u \) from the space \( Y_1 \times \cdots \times Y_k \times Z \) into a Banach space \( W \). Define a new operator \( u_0 \) from the space \( Y_1 \times \cdots \times Y_k \) into the space \( W_0 = L(Z, W) \) by means of the formula

\[
  u_0(y_1, \cdots, y_k)(z) = u(y_1, \cdots, y_k, z) \text{ for } y_i \in Y_i, z \in Z.
\]

It is easy to see that the operator \( u_0 \) is \( k \)-linear and continuous.

Now if

\[
  f_i \in L(v_{ji}, Y_j)
\]

then according to the previous theorem we have

\[
  f = u_0(f_1, \cdots, f_k) \in L(v, W_0).
\]

Define a new operator \( u_1 \) by means of the formula

\[
  u_1(w, z) = w(z) \text{ for } w \in W_0, z \in Z.
\]

**Theorem 7.** If \( \mu \in M(v, Z) \) and \( f = u_0(f_1, \cdots, f_k) \), \( u_0, u \) are defined as above then

\[
  \int u(f_1, \cdots, f_k, d\mu) = \int u_1(f, d\mu).
\]

Now let \( Y_j, Z_j \) \((j = 1, \cdots, k)\) be Banach spaces and let \( (X, V, v) \) be the product of the volume spaces \( (X_j, V_j, v_j) \) \((j = 1, \cdots, k)\). Let

\[
  f_j \in L(v_{ji}, Y_j) \quad \text{and} \quad \mu_j \in M(v_{ji}, Z_j)
\]

Consider a multilinear continuous operator \( u \) from the product of the spaces \( Y_j, Z_j \) \((j = 1, \cdots, k)\) into a Banach space \( W \). Let \( u_0 \) be an operator from the product of the spaces \( Z_j \) \((j = 1, \cdots, k)\) into the space \( W_0 = L(Y_1, \cdots, Y_k; W) \) defined by the formula
for $z_i \in Z_i$, $y_i \in Y_i$.

It is easy to see that the operator $u_0$ is $k$-linear and continuous. Thus from Theorem 4 we get

$$\mu = u_0(\mu_1, \cdots, \mu_k) \in M(v, W_0).$$

Let $u_1$ denote the multilinear continuous operator defined on the space $Y_1 \times \cdots \times Y_k \times W_0$ by means of the formula

$$u_1(y_1, \cdots, y_k, w) = w(y_1, \cdots, y_k) \text{ for } y_j \in Y_j, w \in W_0.$$

We have the following theorem.

**Theorem 8.** If $f_j \in L(v_j, Y_j)$ and $\mu = u_0(\mu_1, \cdots, \mu_k)$, $u_1$ are defined as above, then

$$\int u(f_1, \cdots, f_k, d\mu_1, \cdots, d\mu_k) = \int u_1(f_1, \cdots, f_k, d\mu).$$

The last two theorems allow us to reduce any of the integrals to the following form $\int u(f, d\mu)$. In [5] has been shown how one can reduce the integrals to iterated integrals by means of generalized Fubini’s Theorems.

**References**


