ON THE SPECTRUM OF GENERAL SECOND ORDER OPERATORS

BY M. H. PROTTER AND H. F. WEINBERGER

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Let \( \lambda_1 \) be the lowest eigenvalue of the membrane problem

\[
\Delta u + \lambda u = 0 \quad \text{in } D,
\]

\[
u = 0 \quad \text{on } \partial D.
\]

It was shown by Barta [1] that if \( w > 0 \) in \( D \), then

\[
\lambda_1 \geq \inf \left[ - \frac{\Delta w}{w} \right].
\]

This result has been extended to other selfadjoint problems for second order operators. See [2], [3], and [6].

The purpose of this note is to show that the same technique locates the spectrum of a nonselfadjoint problem in a half-plane. Such a result is of interest in investigating stability, where one needs to know whether there is any spectrum in the half-plane \( \text{Re} \lambda \leq 0 \).

In a bounded domain \( D \) we consider the differential equation

\[
L[u] + \lambda k u = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x) u + \lambda k(x) u
\]

(1)

\[
= -k(x)f(x)
\]

where \( x \sim (x_1, \cdots, x_n) \). The matrix \( a^{ij}(x) \) is symmetric and positive definite, \( k(x) \) is positive, and all the coefficients are real and bounded in \( D \). However, they need not be continuous.

The boundary \( \partial D \) is divided into two disjoint parts \( \Sigma_1 \) and \( \Sigma_2 \), and the boundary conditions are

\[
u = 0 \quad \text{on } \Sigma_1,
\]

(2)

\[
M[u] = \sum_{i=1}^{n} e^i(x) \frac{\partial u}{\partial x_i} + g(x)u = 0 \quad \text{on } \Sigma_2.
\]

The vector field \( e \) points outward from \( D \).

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We shall prove the following theorem about the spectrum of the operator $L$ considered as an operator on the space $C(D)$ of continuous functions with the maximum norm.

**Theorem 1.** Suppose $w(x)$ defined on $D \cup \partial D$ has the properties:
(i) $w(x) > 0$ on $D \cup \partial D$;
(ii) $w \in C^2(D) \cap C^1(D \cup \partial D)$;
(iii) $M[w] \geq 0$ on $\Sigma_2$.

Then the discrete and continuous spectra of the problem (1), (2) are contained in the half-plane

$$\text{Re} \lambda \geq \inf \left(-\frac{L[w]}{kw}\right).$$

**Proof.** Let $\tau = \inf(-L[w]/kw)$, and suppose that $\text{Re} \lambda < \tau$. We wish to show that $\lambda$ is in the resolvent set.

Let $u$ satisfy (1) and (2), and define $v(x) = u(x)/w(x)$.

Substituting $u = vw$ in (1), multiplying the equation by $\bar{v}$, and taking real parts, we obtain

$$\sum_{i=1}^{n} \frac{1}{2} w a^{ij} \frac{\partial^2 |v|^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \frac{1}{2} \left( w b^i + 2 \sum_{j=1}^{n} a^{ij} \frac{\partial w}{\partial x_j} \right) \frac{\partial |v|^2}{\partial x_i}$$

$$+ (L[w] + \text{Re}(\lambda)kw) |v|^2 = \sum_{i=1}^{n} a^{ij} \frac{\partial v}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} - \text{Re}(\bar{v})$$

$$\geq - k \text{Re}(\bar{v}),$$

since $a^{ij}$ is positive definite. The boundary conditions yield

$$|v|^2 = 0 \quad \text{on} \quad \Sigma_1,$$

$$\sum_{i=1}^{n} e^i \frac{\partial |v|^2}{\partial x_i} + 2M[w] |v|^2 = 0 \quad \text{on} \quad \Sigma_2.$$

We observe that $L[w] + \text{Re}(\lambda)kw \leq -(\tau - \text{Re} \lambda)kw$. Therefore by the maximum principle, we find that

$$|v|^2 \leq \frac{1}{\tau - \text{Re} \lambda} \sup_D \frac{\text{Re}(\bar{v})}{w}.$$

Hence
Thus if $\text{Re} \lambda < r$, the operator $L + \lambda k$ has a bounded inverse in the maximum norm on its range. Hence $\lambda$ is in either the residual spectrum or the resolvent set. Therefore the discrete and continuous spectra are contained in the half-plane

$$\text{Re} \lambda \geq \inf_D \left( -\frac{L[w]}{kw} \right)$$

as the theorem states.

In what follows we shall assume that the problem does not have a residual spectrum. That is, we assume that the range of $L + \lambda k$ is dense for some sufficiently small $\lambda$; or, equivalently, that the index is zero.

The following theorem shows that the bound (3) is a lower bound for a real point $\lambda_1$ of the spectrum:

**Theorem 2.** Suppose there is a function $w$ satisfying the conditions of Theorem 1. Then if the spectrum of (1), (2) is not empty, there exists a real number $\lambda_1$ in the spectrum such that the whole spectrum lies in the half-plane

$$\text{Re} \lambda \geq \lambda_1.$$  

**Proof.** Let $\lambda$ be real, and let $v = u/w$, where $u$ is real. Then the problem (1), (2) becomes

$$\sum_i w a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_i \left( w b_i + 2 \sum_j a_{ij} \frac{\partial w}{\partial x_j} \right) \frac{\partial v}{\partial x_i} + (L[w] + \lambda kw)v = -kf \quad \text{in } D,$$

$$v = 0 \quad \text{on } \Sigma_1,$$

$$\sum_i w e_i \frac{\partial v}{\partial x_i} + M[w]v = 0 \quad \text{on } \Sigma_2.$$  

By the maximum principle we see that if $\lambda < \inf(-L[w]/kw)$, then $f > 0$ implies $v > 0$ and hence $u > 0$. Thus the resolvent $R_\lambda$ is positive for $\lambda < \inf(-L[w]/kw)$.
Conversely if \( R_\tau \geq 0 \) for some real number \( \tau \), we find that the solution \( w \) of

\[
L[w] + \tau kw = -k \quad \text{in } D,
\]

\[
\begin{align*}
w &= 1 & \text{on } \Sigma_1, \\
M[w] &= 0 & \text{on } \Sigma_2,
\end{align*}
\]  

is admissible in Theorem 1, so that the spectrum lies in the half-plane \( \Re \lambda \geq \tau \), and \( R_\mu \geq 0 \) for all real \( \mu \leq \tau \).

Now let \( \lambda_1 \) be the limit superior of those \( \lambda \) for which \( R_\lambda \geq 0 \). Then the spectrum is in the half-plane \( \Re \lambda \leq \lambda_1 \). If \( \lambda_1 \) is in the resolvent set, we see by continuity that \( R_{\lambda_1} \geq 0 \). Moreover, for any \( \lambda > \lambda_1 \) with \( \lambda - \lambda_1 < \| R_{\lambda_1} \|^{-1} \) we have \( R_\lambda = (I - (\lambda - \lambda_1)R_{\lambda_1})^{-1}R_{\lambda_1} = R_{\lambda_1} + (\lambda - \lambda_1)R_{\lambda_1}^2 + \cdots \geq 0 \). Thus if \( \lambda_1 \) is in the resolvent set, we obtain a contradic­tion with the definition of \( \lambda_1 \). Hence \( \lambda_1 \) is in the spectrum of (1), (2).

We observe that for any \( \tau < \lambda_1 \) the solution \( w \) of (4) gives the lower bound \( \tau \), so that the lower bound (3) can be made arbitrarily close to \( \lambda_1 \) by a judicious choice of \( w \).

**Remarks**

1. If \( D \) is unbounded but \( \Sigma_2 \) is bounded, we can define a solution of (1), (2) by exhaustion. That is, we obtain the solutions \( u_n \) of

\[
\begin{align*}
L[u_n] + \lambda ku_n &= -kf & \text{in } D \cap \{ |x| < n \}, \\
u_n &= 0 & \text{on } \Sigma_1 \cup \{ |x| = n \}, \\
M[u_n] &= 0 & \text{on } \Sigma_2.
\end{align*}
\]

By the method used in the proof of Theorem 1 we find that if \( \Re \lambda < \inf(-L[w]/kw) \), the functions \( u_n \) converge uniformly to a solution \( u \) of (1), (2). Thus the spectrum still lies in \( \Re \lambda \geq \inf(-L[w]/kw) \).

Theorem 2 can also be extended to this case.

2. If \( D \) and the coefficients of our problem are so smooth that for sufficiently small real \( \mu \) the resolvent \( R_\mu \) is completely continuous in the maximum norm (i.e., the family \( R_\mu [f] \) with \( f \leq 1 \) is equicontinuous), then the spectrum is discrete, so that \( \lambda_1 \) is an eigenvalue.

A theorem of Krein and Rutman [5, Theorem 6.1] shows that the corresponding eigenfunction \( u_1 \) is positive in \( D \). The theorem of Krein and Rutman also states that in this case the adjoint operator \( R_\mu^* \) has the eigenvalue \( (\lambda_1 - \mu)^{-1} \) with a positive eigenfunctional \( u_1^* \). From this fact we can derive Theorem 1 with condition (i) replaced by the weaker condition \( w \geq 0 \). Moreover, we can obtain a complementary upper bound for \( \lambda_1 \):
If \( q(x) \geq 0 \) in \( D \), \( q = 0 \) on \( \Sigma_1 \), and \( M[q] \leq 0 \) on \( \Sigma_2 \), then 
\[ \lambda_1 \leq \sup(-L[q]/kq). \]

3. If the coefficients are so smooth that the adjoint operator \( L^* \) can be formed, and if the boundary conditions are selfadjoint (e.g., \( \Sigma_1 = \partial D \)), an inequality of the same type as (3) may be found by methods of Hooker [4] and Protter [6]. Namely,
\[ \text{Re}(\lambda) \geq \inf_D \left( - \frac{L[w] + L^*[w]}{2kw} \right). \]
This inequality may be stronger or weaker than (3).

**BIBLIOGRAPHY**


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