SOME RESULTS GIVING RATES OF CONVERGENCE IN THE LAW OF LARGE NUMBERS FOR WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES

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Let $\{X_N\}$ for $N=1, 2, \cdots$ be an independent sequence of random variables, and let $S_N=X_1+\cdots+X_N$. Probabilists have expended considerable effort investigating the convergence of $\{(S_N-b_N)/a_N\}$ where $\{b_N\}$ and $\{a_N\}$ are sequences of centering and weighting constants respectively. Recently Baum and Katz in [**1**], [**2**], and [**3**] have investigated the rate of convergence to zero of appropriately normalized sums, obtaining (along with some other results) results on the convergence of series of the form $\sum N^\gamma P\{|S_N-N\mu|>N^\beta\}$ where the X's are assumed to be identically distributed with common mean μ. Pruitt in [**4**] obtained a sufficient condition for sums of the form $S_N=\sum_k a_{N,k}X_k$ to converge to μ.

Now let $\{X_k\}$ be an independent sequence of random variables having finite first moments and define

$$F(y) = \sup_k P\{ | X_k - EX_k | > y\}.$$

Let $\{a_{N,k}\}$ for $N, k=1, 2, \cdots$ be real numbers such that

$$\max_k |a_{N,k}| = CN^{-\beta}, \tag{1}$$

$$\sum_k |a_{N,k}| \leqq CN^{\alpha}, \tag{2}$$

$$\sum_k |a_{N,k}|^t \leqq CN^{-\rho}. \tag{3}$$

Define

$$S_N = \sum_k a_{N,k}(X_k - EX_k).$$

We have obtained the following five theorems.

THEOREM 1. *If $\rho>0$, $\beta>0$, $\alpha<\beta$, $t>1$, and $y^tF(y)\leqq M<\infty$ for all $y>0$, then for every $\epsilon>0$*

$$P\{ |S_N| > \epsilon\} \leqq O(N^{-\rho}).$$

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THEOREM 2. *If* $\rho>0$, $\beta>0$, $\alpha<\beta$, $t>1$, *and* $y^tF(y)\to 0$ *as* $y\to\infty$, *then for every* $\epsilon>0$

$$P\{\,|S_N|\,>\epsilon\}=o(N^{-\rho}).$$

THEOREM 3. *If* $\beta(t-1)-\alpha>0$, $\beta>0$, $\alpha<\beta$, $t>1$, *and* F *satisfies*

$$\lim_{y\to\infty}F(y)=0\quad \textit{and}\quad \int y^t\,|\,dF(y)\,|\,<\infty,$$

then for every $\epsilon>0$

$$\sum_N N^{\beta(t-1)-\alpha-1}P\{\,|S_N|\,>\epsilon\}<\infty.$$

THEOREM 4. *If* $\rho>0$, $\beta>0$, $\alpha<\beta$, $t\geqq 1$, *and there exists a nonnegative and nonincreasing real-valued function* $G(x)\geqq F(x)$ *satisfying*

$$\lim_{y\to\infty}G(y)=0\quad \textit{and}\quad \int_0^\infty y^t\,|\,dG(y)\,|\,<\infty$$

such that

$$\sup_{x\geqq 1}\ \sup_{y\geqq x}\frac{y^tF(y)}{x^tG(x)}=\gamma<\infty, \tag{4}$$

then for every $\epsilon>0$

$$\sum_N N^{\rho-1}P\{\,|S_N|\,>\epsilon\}<\infty. \tag{5}$$

THEOREM 5. *If* $\rho>0$, $\beta>0$, $\alpha<\beta$, $t\geqq 1$, *and* F *satisfies*

$$\lim_{y\to\infty}F(y)=0\quad \textit{and}\quad \int_0^\infty y^t\log^+y\,|\,dF(y)\,|\,<\infty,$$

then (5) *holds for every* $\epsilon>0$.

One should immediately notice that for $t\geqq 1$ we have

$$\sum_k |a_{N,k}|^t\leqq\left(\max_k|a_{N,k}|\right)^{t-1}\sum_k|a_{N,k}|$$
$$\leqq C^2N^{\alpha-\beta(t-1)}$$

so that ρ can be assumed to be at least as large as $\beta(t-1)-\alpha$. Note that Theorem 1 implies $\sum_N N^{\rho-1-\delta}P\{|S_N|>\epsilon\}<\infty$ for every $\delta>0$ so that the additional assumptions used in Theorems 4 and 5 do not give very much more than that already obtained in Theorem 1. The assumption (4) can readily be violated but most "reasonable" distributions will satisfy it.

Though Theorems 4 and 5 are considerably stronger than Theorem 3, two known results can be obtained as corollaries of Theorem 3 by specializing the constant t and the constants α and β from (1) and (2). Theorem 2 of [**4**] is obtained by setting $t=1+1/\gamma$, $\alpha=0$, and $\beta=\gamma$; a part of Theorem 3 of [**2**] and [**3**] is obtained by leaving t as is and setting $\alpha=1-r/t$ and $\beta=r/t$ with $\frac{1}{2}<r/t\leqq 1$.

The motivation for this work was as follows. The average $\frac{1}{2}X_1+\frac{1}{4}X_2+\frac{1}{4}X_3$ is at least as "fine" an average as is $\frac{1}{2}X_1+\frac{1}{2}X_2$ and in some sense the first average is "finer" than the second. It has always seemed reasonable to the authors that a finer average than the standard average $(1/N)\{X_1+\cdots+X_N\}$ should not hurt convergence any and might actually improve the rate of convergence if one could use the right quantitative measure of the improvement in averaging. The exponent ρ used in (3) seems to be the correct measure of averaging to use.

The methods used in the proof of these theorems apparently originated with Erdös [**5**]. The method was modified and improved by Katz [**1**] and modified still more by Pruitt [**4**]. Detailed proofs will appear elsewhere.

References

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