LOCALLY FLAT NONEMBEDDABILITY OF CERTAIN PARALLELIZABLE MANIFOLDS

BY WU-CHUNG HSIEANG

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1. Introduction and statement of result. This note is a supplement to the joint papers of R. H. Szczarba and the author [6], [7]. We proved in [6], [7] that for any positive integer \( q > 1 \), there is a differentiable parallelizable manifold \( M_q \) of dimension \((2^{4q+1} - 8q - 2)\) which can be differentiably immersed in Euclidean space of codimension 1 but can not be differentiably embedded in Euclidean space of codimension \( 8q \). As a consequence, the dimension difference of the best differentiable immersion and the best differentiable embedding in Euclidean space can be arbitrarily large. One may ask the same type of question for topological or combinatorial immersion and embedding. In this note, we shall modify the argument of [6], [7] to show that \( M_q (q > 1) \) actually has no locally flat topological (hence no combinatorial) embedding in Euclidean space of codimension \( 8q \). Since we used the normal bundle of a differentiable embedding and Adams' solution of vector field problem [1] in the original proof of [6], [7], differentiability seemed to be essential. However, we shall replace the normal bundle by the normal fibre space of Nash-Fadell-Spivak [9], [4], [10] and use a corollary of Adams' solution of vector field problem that \([\tau_{2q-1}, \tau_{2q-1}]\) is not an \((8q+1)\)-fold suspension to show the locally flat nonembeddability of \( M_q (q > 1) \) in Euclidean space of codimension \( 8q \). The author is indebted to Professor John Milnor for his comments.

Let us first recall the manifolds \( M_q (q \geq 1) \). If \( \xi \) and \( \eta \) are sphere bundles with a common base, we use \( \xi \oplus \eta \) to denote the vector bundle associated with \( \xi \) and \( \xi \ast \eta \) the sphere bundle associated with \( \xi \oplus \eta \).

Let \( S^{n-1} \) be the \((n-1)\)-sphere where \( n = 2^{4q}, q \geq 1 \). It follows from results of Eckmann [3] and Adams [1] that \( S^{n-1} \) has exactly \( 8q \) independent vector fields. Thus we can find an \((n-8q-1)\)-sphere bundle \( \xi_q \) over \( S^{n-1} \) with a cross section and with the property that \( \xi_q \ast \theta^{8q-1} = \tau(S^{n-1}) \), the tangent sphere bundle of \( S^{n-1} \). (Here \( \theta \) denotes the trivial \((r-1)\)-sphere bundle.) Let \( M_q \) be the total space of \( \xi_q \).

Let \( M^n \) and \( N^m \) be two topological manifolds. A topological embedding (immersion) \( f: M^n \to N^m \) is said to be locally flat [4], [5], if...
for each point \( x \in M \), there are neighborhoods \( U(x) \) and \( V(f(x)) \) such that \((V(f(x)), f(U(x)))\) is topologically equivalent to \((R^m, R^n)\).

Clearly, a differentiable embedding is locally flat. It follows from Zeeman's unknotting theorem that a combinatorial embedding of codimension greater than 2 is also locally flat [13]. Moreover, local flatness is independent of the differential or combinatorial structure.

**Theorem.** For \( q > 1 \), \( M_q \) has no locally flat embedding in Euclidean space with codimension \( 8q \).

**Remark 1.** The 22-dimensional manifold \( M_1 \) has no locally flat embedding in \( R^{28} \).

**Remark 2.** Let \( N_q \) be a combinatorial manifold of the homotopy type of \( M_q \) \((q > 1)\). By a recent result of Irwin (Ann. of Math. 82 (1965), 1–14), there is a combinatorial embedding \( s: S^{n-1} \subset N_q \) which represents a generator of \( H_{n-1}(N_q; \mathbb{Z}) \). Let \( U \) be a neighborhood of \( s(S^{n-1}) \) in \( N_q \). One can actually show that \( U \) has no locally flat embedding in Euclidean space of codimension \( 8q \). 

**Corollary 1.** For any integer \( k \), there are manifolds which have locally flat immersion in Euclidean space with codimension 1 but have no locally flat embedding in Euclidean space with codimension \( k \).

**Corollary 2.** For any integer \( k \), there are combinatorial manifolds which can be combinatorially immersed in Euclidean space with codimension 1 but cannot be combinatorially embedded in Euclidean space with codimension \( k \).

**Proof.** Consider the \( C^r \)-triangulation \((r \geq 1)\) of the differentiable manifolds \( M_q \) \((q > 1)\). Following from Zeeman [13], every combinatorial embedding with codimension greater than 2 is locally flat. Corollary 2 follows immediately from the theorem.

2. **Spherical normal fibre space.** Let \( f: M^n \subset N^m \) be a locally flat embedding. Following [9], [4], [10], we consider the path space 

\[
E = \{ w \mid w \in (N^m)^1, w(t) \in M^n \text{ if and only if } t = 0 \}.
\]

The initial projection \( p: E \to M^n \) defined by \( p(w) = w(0) \) is an \((m - n - 1)\)-spherical fibre space over \( M \),

\[
\nu: S^{m-n-1} \to E \to M^n.
\]

\( \nu \) is called the spherical normal fibre space of the embedding \( f \). If \( f \) is a differentiable embedding with differentiable normal sphere bundle \( \eta \), then \( \eta \) is fibre homotopically equivalent to the spherical normal fibre space \( \nu \) when \( f \) is considered as a locally flat embedding [4].
Lemma 1 (Massey [8], Stallings [11]). Let \( f: S^n \subseteq S^{n+k} \) (\( n \geq 2, k \geq 3 \)) be a locally flat embedding. Then, the spherical normal fibre space \( \nu \) of \( f \) is fibre homotopically trivial.

Proof. By Stallings [11], \( f(S^n) \subseteq S^{n+k} \) is unknotted. Hence, \( S^{n+k} - f(S^n) \) is of the homotopy type of \( S^{k-1} \). Then, \( \nu \) is fibre homotopically trivial by the argument of [8].

Let \( \xi_1: S^{k-1} \rightarrow E_1 \rightarrow \nu_1 X \), and \( \xi_2: S^{l-1} \rightarrow E_2 \rightarrow \nu_2 X \) be two spherical fibre spaces. Let \( E \) be the subset of the join \( E_1 \ast E_2 \) of \( E_1 \) and \( E_2 \), consisting of the points \((x_1, t, x_2)\) with the property that \( p_1 x_1 = p_2 x_2 \) where \((x_1, t, x_2) \subseteq E_1 \ast E_2 \). The projection \( q: E \rightarrow X \) defined by

\[
q(x_1, t, x_2) = p_1(x_1) = p_2(x_2)
\]

is a spherical fibre space \( \xi \) called the Whitney join of \( \nu_1 \) and \( \nu_2 \), and will be denoted by \( \xi_1 \ast \xi_2 \).

Lemma 2 (Spivak [10]). Let \( f_1: M^n \rightarrow N^n_1 \) be a differentiable embedding with the normal sphere bundle \( \nu_1 \) and let \( f_2: N^n_1 \rightarrow N^n_2 \) be a locally flat embedding with spherical normal fibre space \( \nu_2 \). Then, the spherical normal fibre space \( \nu \) of the composite embedding

\[
f_2 f_1: M^n \rightarrow N^n_1 \rightarrow N^n_2
\]

(which is clearly locally flat) is fibre homotopically equivalent to \( \nu_1 \ast (\nu_2| M^n) \).

Proof. Let \( D \) be a closed tubular neighborhood of \( f(M^n) \) in \( N^n_1 \) such that the exponential map (under some proper Riemannian metric of \( N^n_1 \)) maps the closed disc bundle

\[
\tilde{\nu}_1: D^{n-n} \rightarrow E_1 \rightarrow M^n
\]

associated with \( \nu_1 \) diffeomorphically onto \( D \). We identify \( D \) with \( E_1 \).

Since \( M^n \) is a deformation retract of \( D \), the fibre space \( \tilde{p}_1(\nu_2| M^n) \) is fibre homotopically equivalent to \( \nu_2| D \). Let \( g: \tilde{p}_1(\nu_2| M^n) \rightarrow \nu_2| D \) be a fibre homotopical equivalence. Let us construct a fibre map

\[
h: \nu_1 \ast (\nu_2| M^n) \rightarrow \nu
\]

as follows. First recall that (i) every point in \( D \) is of the form \( \exp(x_1) \) with a vector \( x_1 \) of length \( \leq 1 \) and normal to \( M^n \) (with respect to some proper Riemannian metric of \( N^n_1 \)), (ii) every point in \( (\nu_2| M^n) \) is a path in \( N^n_2 \) issuing from \( M^n \) and never touching \( N^n_1 \) again, (iii) every point of \( \nu_1 \ast (\nu_2| M^n) \) is of the form

\[
(\exp(x_1), t, x_2) \text{ for } 0 \leq t \leq 1.
\]
Define
\[ h(\exp(x_1), t, x_2) = \exp(ux_1) \quad \text{for } 0 \leq u \leq t, \]
\[ = g[p_1 \exp(x_1)(x_2)](u - t) \quad \text{for } t \leq u \leq 1. \]

Using the differentiability of \( f_1 \) and the local flatness of \( f_2 \), it is easy
to check that \( h \) induces homotopical equivalence on each fibre. By 
A. Dold's criterion [2], \( h \) is a fibre homotopical equivalence.

**Remark.** A similar statement of Lemma 2 was proved in [10]. Since [10] is still unpublished and the case which we need is rather
special, we include a somewhat simpler proof for completeness.

3. **Proof of the theorem.** Let \( G_k \) be the \( H \)-space of degree 1 maps of
\( S^{k-1} \rightarrow S^{k-1} \). By [2], [11], the fibre homotopical equivalence classes
of \( S^{k-1} \)-fibre spaces over \( S^p \) are one-one correspondence to the elements
of \( \pi_{p-1} (G_k) \). Let \( \sigma^{m \cdot k} : G_k \rightarrow G_m \) \((m \geq k)\) be the obvious inclusion
induced by the suspensions of the degree 1 maps of \( S^{k-1} \). Let \( \nu_1, \nu_2 \)
be \( S^{k-1} \)-fibre spaces over \( S^p \) corresponding to \( \pi_1 \subset \pi_{p-1} (G_k) \),
\( \pi_2 \subset \pi_{p-1} (G_1) \) respectively. By the argument of Lemma 3.1 of [6], we
see that \( \nu_1 \ast \nu_2 \) corresponds to

\[ \sigma^1 (\nu_1) + \sigma^k (\nu_2) \]
in \( \pi_{p-1} (G_{k+1}) \).

Now, suppose that \( M_q \) \((q > 1)\) has a locally flat embedding in Euclidian space of codimension \( 8q \) with \( \nu \) as the spherical normal fibre
space. We choose a differentiable cross section \( s : S^{n-1} \rightarrow M_q \) \((M_q \) is
given the natural differentiable structure\) of \( \xi \) which has a normal sphere bundle \( \nu_8 \) with the property \( \nu_8 \ast \theta^1 = \xi \). By Lemma 1, the com-
posite embedding

\[ S^{n-1} \rightarrow M_q \rightarrow \mathbb{R}^{2n-2} \]

has a fibre homotopically trivial spherical normal fibre space. Hence, \( \nu_8 \ast \nu \) \((\nu \mid S^{n-1}) \) is fibre homotopically trivial by Lemma 2. Let
\( \tilde{\nu} \in \pi_{n-2} (G_{8q}) \), \( \tilde{\nu} \in \pi_{n-2} (G_{n-8q-1}) \), and \( \tilde{\tau} \in \pi_{n-2} (G_{n-1}) \) be the elements
corresponding to \((\nu \mid S^{n-1}) \), \( \nu_8 \), and \( \tau (S^{n-1}) \) respectively. Since
\( \nu_8 \ast (\nu \mid S^{n-1}) \) is fibre homotopically trivial,

\[ \sigma^8 \tilde{\nu} + \sigma^{n-8q-1} (\tilde{\nu}) = 0. \]

But \( \nu_8 \ast \theta^1 \ast \theta^{8q-1} = \xi \ast \theta^{8q-1} = \tau (S^{n-1}) \). So

\[ \sigma^8 \tilde{\nu} = \tilde{\tau}. \]

Thus \( 1 \) and \( 2 \) show that \( \tilde{\tau} \) is an \((n-8q-1)\)-fold suspension. Now
let $J : \pi_{n-1}(G_k) \to \pi_{k+n-1}(S^8)$ be defined in the usual way. Then the element

$$[\iota_{n-1}, \iota_{n-1}] = J(\bar{r}) \subseteq \pi_{2n-2}(S^{n-1})$$

must be an $(n - 8q - 1)$-fold suspension. But according to [1, Corollary 1.3], this element $[\iota_{n-1}, \iota_{n-1}]$ is not even an $(8q + 1)$-fold suspension. Since

$$n - 8q - 1 \geq 8q + 1$$

for $q > 1$, this gives a contradiction, and completes the proof.

**References**