Let $A$ be an algebra of continuous complex valued functions on a compact Hausdorff space $X$. If $A$ is uniformly closed, contains the constant functions and separates the points of $X$, then we call $A$ a function algebra on $X$. $M(A)$, the space of all multiplicative linear functionals on $A$, is a compact Hausdorff space containing a homeomorphic image of $X$ and $A$ can be viewed as a function algebra on $M(A)$. In [1] Gleason observed that the relation $x \sim y$ if and only if $\|x-y\|_A < 2$ is an equivalence relation on $M(A)$. The equivalence classes are called the "parts" of $M(A)$.

Hoffman has asked if parts must be connected. The purpose of this note is to exhibit some disconnected parts.

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**Theorem 1.** There is a function algebra $A$ with two generators such that $M(A)$ contains a part homeomorphic to two copies of the open unit disc.

**Theorem 2.** Let $K$ be a compact Hausdorff space. Then there exists a function algebra $A$ such that $M(A)$ contains a part homeomorphic to $K$. Every continuous function on that part is the restriction of a function in $A$.

The last assertion implies that no subset of the part can be given any analytic structure. We will prove Theorem 1 here. The proof of Theorem 2 is more complicated and will be presented elsewhere.

The disc algebra is the algebra of all functions continuous on the closed unit disc $D$ and analytic on the interior. $T^1$ denotes the unit circle in the complex plane and $T^2$ denotes the torus $T^1 \times T^1$.

**Proof of Theorem 1.** Pick a function $h_1$ in the disc algebra mapping $D$ onto $\{z \in \mathbb{C} : |z| \leq 1, |z-1| \leq 3/4 \}$. Let $h_2 = -h_1$. Choose an open arc $I$ on $T^1$ such that $|h_1| = 1$ on $I$ and set

$$X_j = \{(z, w) \in T^2 : z \in I, w = h_j(z)\} \quad (j = 1, 2),$$

$$Y = \{(z, w) \in T^2 : z \notin I\}.$$
Then \( X = X_1 \cup X_2 \cup Y \) is a compact subset of \( T^2 \). Our algebra \( A \) is \( P(X) \), the uniform closure on \( X \) of the polynomials in the coordinate functions \( z \) and \( w \). \( M(A) \) is the polynomial hull of \( X \),

\[
\hat{X} = \left\{ p \in C^2 : |f(p)| \leq \max_{x \in X} |f(x)| , \text{ for every polynomial, } f \right\}.
\]

Set

\[
V = \{(z, w) \in \hat{X} : |z| < 1\},
\]

\[
V_j = \{(z, w) \in C^2 : |z| < 1, w = h_j(z)\} \quad (j = 1, 2).
\]

We show that \( V = V_1 \cup V_2 \) and that \( V \) is a part of \( X \). The proof is in two steps.

(i) \( V \) is the disjoint union \( V_1 \cup V_2 \).

Since \( f(z, h_j(z)) \) is in the disc algebra for any polynomial \( f \) and \( \{(z, h_j(z)) : |z| = 1\} \subset X_j \cup \{q\} \subset \hat{X} \), we have \( V_j \subset V \). Because \( h_1 \) never vanishes, \( V_1 \cap V_2 = \emptyset \).

Take \( q = (s^0, w^0) \in V \) and let \( m \) be a representing measure on \( X \) for \( q \). Let \( \mu \) be the measure

\[
\mu = (w - h_1(z))(w - h_2(z))(z - z^0)m.
\]

Then \( \mu \) is supported by \( Y \) and annihilates \( A \) because the multiplicative measure \( m \) has been multiplied by a function in \( A \) vanishing on \( X_1 \cup X_2 \cup \{q\} \). Hence \( \mu \) is orthogonal to \( P(Y) \). Now the restricted function \( (z - z^0)Y \) is invertible in \( P(Y) \) so that the measure

\[
\frac{1}{z - z^0} \mu = (w - h_1(z))(w - h_2(z))m
\]

annihilates all polynomials. But then

\[
0 = \int 1(w - h_1(z))(w - h_2(z)) \, dm = (w^0 - h_1(z^0))(w^0 - h_2(z^0))
\]

because \( m \) represents \( q \). Thus \( w^0 = h_j(z^0) \) for some \( j \) and \( q \in V_j \).

(ii) \( V \) is a part of \( M(A) \).

It is clear that no point in \( V \) is equivalent to any point in \( X \setminus V \) as the function \( z \) has modulus one on \( X \setminus V \). And it is seen directly that any two points in \( V_j \) are equivalent. Let \( p_j = (0, h_j(0)) \in V_j \). The proof will be complete when we show \( p_1 \sim p_2 \).

Notice that for \( f \in A \) we have

\[
f(p) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}, h_j(e^{i\theta})) \, d\theta.
\]
Choose an open arc $J$ on $T^1$ such that $|h_j| \leq \frac{1}{2}$ on $J$. Let $f \in A$, $|f| \leq 1$. Then

$$|f(p_1) - f(p_2)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}, h_1(e^{i\theta})) - f(e^{i\theta}, h_2(e^{i\theta}))| \, d\theta$$

$$\leq \frac{1}{2\pi} \int_{\pi \setminus J} 2d\theta + \frac{1}{2\pi} \int_J |f(e^{i\theta}, h_1(e^{i\theta})) - f(e^{i\theta}, h_2(e^{i\theta}))| \, d\theta.$$ 

Recall that there is a constant $c < 2$ such that whenever $|z_1| \leq \frac{1}{2}$, $|z_2| \leq \frac{1}{2}$ and $g$ is in the disc algebra $|g(z_1) - g(z_2)| < c ||g||$. Now for $e^{i\theta} \in J$, $w \rightarrow f(e^{i\theta}, w)$ is a function in the disc algebra of norm at most one. Since $|h_j(e^{i\theta})| \leq \frac{1}{2}$ for $e^{i\theta} \in J$, we therefore have

$$|f(e^{i\theta}, h_1(e^{i\theta})) - f(e^{i\theta}, h_2(e^{i\theta}))| < c$$

uniformly on $J$. Hence

$$|f(p_1) - f(p_2)| \leq \frac{1}{2\pi} \int_{\pi \setminus J} 2d\theta + \frac{1}{2\pi} \int_J cd\theta = c' < 2.$$ 

Consequently, $p_1 \sim p_2$ and the proof is complete.

*Added in proof.* The author has recently improved Theorem 2 as follows.

**Theorem 3.** Let $K$ be a topological space. Then $K$ is homeomorphic to some Gleason part if and only if $K$ is completely regular and σ-compact.

**Reference**


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