A CHARACTERIZATION OF Q-DOMAINS

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Let $R$ be an integral domain with quotient field $K$. By an overring of $R$ is meant a ring $B$ with $R \subseteq B \subseteq K$. $R$ is a Q-domain if every overring of $R$ is a ring of quotients of $R$ with respect to some multiplicative system in $R$. A P-domain is a Prüfer ring. Q-domains have been investigated by Gilmer and Ohm [3] and by Davis [2]. All Q-domains are P-domains, and a long list of characterizations of P-domains is available in Bourbaki [1, pp. 93–94]. Noetherian Q-domains are characterized in [3] as those Dedekind domains whose ideal class group is a torsion group. The purpose of this paper is to obtain a characterization of general Q-domains (Theorem 5).

Let $K^*$ denote the set of nonzero elements of $K$. If $x \in K^*$, we define the numerator ideal of $x$ to be $N(x) = \{a \in R: a = bx, \text{ for some } b \in R\}$ and the denominator ideal of $x$ to be $D(x) = \{b \in R: bx \in R\}$. Since $N(x) = Rx \cap R$ and $D(x) = N(1/x)$, $N(x)$ and $D(x)$ are ideals in $R$.

If $P$ is a prime ideal in $R$, $R_P$ denotes the local ring of $R$ at $P$.

**Theorem 1.** $R$ is a P-domain if and only if $N(x) + D(x) = R$, for all $x \in K^*$.

**Proof.** First note that for any prime ideal $P \subseteq R$, $x \in R_P$ if and only if $D(x) \subseteq P$, and hence $1/x \in R_P$ if and only if $N(x) \subseteq P$. Therefore $R_P$ is a valuation ring if and only if $N(x) \subseteq P$ or $D(x) \subseteq P$, for all $x \in K^*$, i.e., if and only if $N(x) + D(x) \subseteq P$, for all $x \in K^*$. Thus to say the ideals $N(x) + D(x)$ are all improper is equivalent to saying all the local rings $R_P$ are valuation rings, i.e., $R$ is a P-domain.

**Corollary 2.** If $R$ is a P-domain and $x \in K^*$, then the numerator and denominator ideals of $x$ can be generated by two elements.

**Proof.** Since $N(x) + D(x) = R$ we can write $x = a/b = a'/b'$, where $a + b' = 1$. Then $D(x) = (b, b')$, for $c \in D(x)$ implies $c = ca + cb' = cxb + cb'$, with $cx \in R$. Also $N(x) = D(1/x) = (a, a')$.

In order to prove Theorem 5, we need to make two remarks concerning P-domains.

**Remark 3.** If $R$ is a P-domain, then the finitely generated fractionary ideals of $R$ form a group [1]. Moreover, if $A = (a_1, \ldots, a_n)$

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is a finitely generated ideal, then the inverse of $A$ is given by $A^{-1} = R:A = \{ x \in K \mid xA \subseteq R \}$. Equivalently, $A^{-1} = (b_1, \ldots, b_n)$, where $a_i b_i \in R$ and $\sum a_i b_i = 1$ [4, pp. 271–272].

**Remark 4.** If $R$ is a $P$-domain, and $B$ is any overring of $R$, then $B$ is the intersection of all the local rings $R_P$ of $R$ which contain $B$ (Proposition 2 of [2]).

**Theorem 5.** Let $R$ be a $P$-domain. Then $R$ is a Q-domain if and only if for every finitely generated ideal $A \subseteq R$, there is an element $f \in R$ such that $\sqrt{A} = \sqrt{(f)}$.

**Proof.** First let $R$ be a Q-domain. We follow the argument of Theorem 2.5(g) of [3]. Let $A = (a_1, \ldots, a_n)$ be a finitely generated ideal in $R$. Let $B$ be the $A$-transform of $R$, i.e., $B = \{ x \in K \mid xA^n \subseteq R \text{ for some } n \geq 0 \}$. By Remark 3, we may write $A^{-1} = (b_1, \ldots, b_n)$ with $\sum a_i b_i = 1$, and then $B = \bigcup (A^{-1})^n = \bigcup (R[b_1, \ldots, b_n])$. Now $B$ is a ring of quotients of $R$ with respect to some multiplicative system $S$. Thus there is $f \in S$ such that $b_i = c_i f$, $1 \leq i \leq n$, with $c_i \in R$. Then $f = f(\sum a_i b_i) = \sum c_i a_i \in A$. Moreover, $f \in S$ implies $1/f \in B$, so $(1/f)(A^n) \subseteq R$, for some $n$, i.e., $A^n \subseteq (f)$. Since $A^n \subseteq (f) \subseteq A$, it follows that $\sqrt{A} = \sqrt{(f)}$.

Conversely suppose $R$ satisfies the condition stated in the theorem. To prove $R$ is a Q-domain it suffices to prove $R[x]$ is a ring of quotients of $R$, for every $x \in K^*$, by Proposition 1.4 of [3]. Let $x \in K^*$. Then, by Corollary 2, $D(x)$ is finitely generated, so $\sqrt{D(x)} = \sqrt{(f)}$, for some $f \in R$. Hence if $P$ is any prime ideal in $R$ we have $(f) \subseteq P$ if and only if $D(x) \subseteq P$, and thus $R[1/f] \subseteq R_P \iff 1/f \in R_P \iff f \in P \iff D(x) \subseteq P \iff x \in R_P \iff R[x] \subseteq R_P$. It follows by Remark 4 that $R[x] = R[1/f]$ which is a ring of quotients of $R$. This completes the proof of Theorem 5.

It should be noted that Theorem 5 does not answer the question raised on [3], namely: is the ideal class group of every Q-domain a torsion group?

**References**