SOLVABILITY OF THE FIRST COUSIN PROBLEM AND VANISHING OF HIGHER COHOMOLOGY GROUPS FOR DOMAINS WHICH ARE NOT DOMAINS OF HOLOMORPHY. II

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This work is a continuation of [2]. In [2] we studied the cohomology groups $H^q(X \setminus A, \emptyset)$ where $A(\subset X)$ is a closed generalized polydisc. Here we consider the general case where $A$ is the closure of a domain of holomorphy. This general case was treated in [1] for $q=1$, but the present method (for $q \geq 1$) is entirely different.

We adopt the definition in [4] of analytic polyhedron. By an analytic polyhedron in general position we mean an analytic polyhedron as defined in [3, p. 288].

**Theorem 1.** Let $A \subset \mathbb{C}^n$ be the closure of a bounded analytic polyhedron in general position and let $X$ be any open set in $\mathbb{C}^n$, containing $A$. Then the restriction map

$$H^q(X, \emptyset) \rightarrow H^q(X \setminus A, \emptyset) \quad (1 \leq q \leq n - 2)$$

is bijective.

We proceed as in [2] except that now we take $G = B \setminus A$ where $B = \{z \in D; f_j(z) \in \Delta'_j \text{ for } j = 1, \ldots, N\}$ where $A$ is defined by $A = \{z \in D; f_j(z) \in \Delta_j \text{ for } j = 1, \ldots, N\}$ where $f_j$ are holomorphic in $D$, $\Delta'_j$ is some open neighborhood of $\Delta_j$, and $B \subset D$. (The argument in [2] can be simplified by dropping out the sets $U_{i_1}, \ldots, U_{i_q}$ which occur in the covering $X \setminus A$.) All we need to prove is the following lemma.

**Lemma.** $H^p(G, \emptyset) = 0$ for $1 \leq p \leq n-2$.

**Proof.** For simplicity we take $\Delta_j$ to be the unit disc and $\Delta'_j$ to be a disc with radius $1 + \epsilon$, homothetic to $\Delta_j$. Clearly $G = \bigcup_{i=1}^{N} U_i$ where $U_i$ is defined as $B$ except for the additional condition $|f_i(z)| > 1$. Thus, each $U_i$ is also an analytic polyhedron. We next proceed analogously to [6, p. 349] and represent $f_{i_1}, \ldots, f_{i_p}$ in $U = \bigcap_{i=1}^{N} U_i$ as $\sum C_M(f_{i_1}, \ldots, f_{i_p})$ where $M = \{M', M''\}$ is a set of indices $j_1, \ldots, j_n$ such that the integration in $C_M(f)$ is taken over $|f_j| = \gamma_j, \ldots, |f_{j_n}| = \gamma_n$ where $\gamma_j = 1$ if $j \in M''$ and $\gamma_j = 1 + \epsilon$ if $j \in M'$; the above integral representation is that given by the Cauchy-Weil formula [3].

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Actually we should have used the representation in compact subsets of $U$, but since this does not affect all the arguments below, we simplify the notation by representing $f_{i_0} \ldots i_p$ in $U$.

One verifies that (a) if $i \in M''$, $i \notin \{i_0, \ldots, i_p\}$ then $C_M(f_{i_0} \ldots i_p) = 0$.

Indeed this follows by the Cauchy-Poincaré theorem [3, p. 264] applied in the $(n+1)$-dimensional set defined by $|f_j| = 1 + \epsilon$ for $j \in M'$, $|f_j| = 1$ for $j \in M'' \setminus \{i\}$, and $|f_i| \leq 1$. Since $p + 1 < n$, it follows from (a) that (b) $C_M(f_{i_0} \ldots i_p) = 0$ if $M'' = \{1, 2, \ldots, n\}$. Next, (γ) $C_M(f_{i_0} \ldots i_p)$ is holomorphic in $U_{i_0} \ldots i_p = \bigcap_{j=0}^{p} U_{i_j}$, since, by (α), we may assume that $M'' \subset \{i_0, \ldots, i_p\}$. Finally, (δ) if $i \in M''$ then $C_M(f_{i_0} \ldots i_{p-1})$ is holomorphic in $U_{i_0} \ldots i_{p-1}$. To construct $g$ with $\delta g = f$ (for any given $p$-cocycle) it suffices to construct, for each fixed $M$, $g$ with $\delta g = C_M(f)$.

We may assume that there is an $i \in M''$, $1 \leq i \leq n$, since otherwise $C_M(f) = 0$ by (β). We then take $g_{i_0} \ldots i_{p-1} = C_M(f_{i_0} \ldots i_{p-1})$.

**Corollary 1.** If $N = n$ in Theorem 1 then there is a surjective map (1) also for $q = n$. If, further, $X$ is a domain of holomorphy, then $H^{n-1}(X \setminus A, \emptyset) \neq 0$.

The first part follows by observing that $H^n(G, \emptyset) = 0$ and using the Mayer-Vietoris sequence (see [0, p. 236]) for $B, X \setminus A$. If the second part is false then $H^q(X \setminus A, \emptyset) = 0$ for $1 \leq q \leq n$. Employing Dolbeault's theorem and [5, Theorem 4.2.9] it follows that $X \setminus A$ is a domain of holomorphy.

**Theorem 2.** Let $A = \cap_{j=1}^n X_j$ where $X_{j-1} \supset X_j$, $X_j$ is a bounded domain of holomorphy in $\mathbb{C}^n$ and $A$ is a closed set, and let $X$ be any open set in $\mathbb{C}^n$, containing $A$. Then the restriction map (1) is bijective for $1 \leq q \leq n - 2$ and injective for $q = n$.

**Proof.** Each $X_j$ can be exhausted by a sequence of analytic polyhedra with $N = n$ (see [4, p. 218]), and by slightly modifying the domains in which the values of the functions (defining the analytic polyhedron) lie, we get a sequence of analytic polyhedra in general position. Thus we can write $A = \cap_{j=1}^n P_j$ where $P_j$ are analytic polyhedra in general position, and $P_{j-1} \supset P_j$. Take a covering $W$ of $X \setminus A$ by domains of holomorphy such that for each $j = 1, 2, \ldots$, there is a subset of $W$ which is a covering of $X \setminus P_j$ and such that the closure of each set in $W$ does not intersect $\partial A$. Using Leray's lemma [4] and the fact that the restriction maps

$$H^r(X, \emptyset) \to H^r(X \setminus \overline{P}_j, \emptyset) \quad (r = q, q - 1)$$

are bijective (for any $1 \leq q \leq n - 2$) it follows by the isomorphisms $H^r(X \setminus \overline{P}_j, \emptyset) \to (H^r(X \setminus \overline{P}_{j-1}, \emptyset))$ and [0, p. 241 and p. 250] that the map (1) is bijective we actually need only the injectivity of (2) for $r = q$ and the surjectivity of (2) for $r = q - 1$. 

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Example. If $A$ is a compact convex set, or if $\partial A$ is $C^2$ and strictly pseudoconvex, then $A$ satisfies the assumptions in Theorem 2. If, in particular, $X$ is a domain of holomorphy, then $H^q(X \setminus A, \mathfrak{g}) = 0$ for $1 \leq q \leq n-2$.

Added in proof. Theorems 1, 2 remain true if $\emptyset$ is replaced by any coherent analytic sheaf $\mathcal{F}$ over $x$, free in a neighborhood of $A$. Assume now that $\mathcal{F}$ has a free resolution of length $d$ in a neighborhood of $\partial A$. Then the lemma holds for $1 \leq p \leq n-2-d$. Using a covering of $x \setminus A$ as in [2] and, additionally, a domain of holomorphy $U^*$ containing $A$ but not intersecting the $U_j$ for $j = i_0, \cdots, i_q$, we get:

**Theorem 3.** If $A, X$ are as in Theorem 2 and if $\mathcal{F}$ is as above, then the restriction map (1), with $\emptyset$ replaced by $\mathcal{F}$, is bijective for $2 \leq q \leq n-2-d$.

This theorem yields the following result on cohomology with compact support: $H^q\left(\Omega, \mathcal{F}\right) = 0$ for $2 \leq q \leq n-1-d$, if $\Omega$ is a domain of holomorphy in $C^n$. (Overlapping results were proved, by a different method, in [0], using Serre’s duality theorem.)

**Proof for $\mathcal{F} = \emptyset$:** Given a $\partial$-closed $q$-form $f$, with compact support in $\Omega$, solve $\bar{\partial}g = f$ in $C^n$; then solve $\bar{\partial}v = g$ outside some compact analytic polyhedron in $\Omega$ (using Theorem 3). $u = v - \bar{\partial}(\bar{\partial}v)$, for some $\xi \in C_0^\infty$ satisfies $\partial u = f$ and has compact support in $\Omega$. For general $\mathcal{F}$ we work with $q$-cochains and coboundary operators. The above proof, together with $H^q\left(\Omega, \emptyset\right) \neq 0$, leads to:

**Corollary.** If $\Omega$ is a domain of holomorphy and a star domain, and if $B$ is any open set with $\Omega \subset \subset G$, then $H^{n-1}(B \setminus \Omega, \emptyset) \neq 0$.

**References**


2. ———, Solvability of the first Cousin problem and vanishing of higher cohomology groups for domains which are not domains of holomorphy, Bull. Amer. Math. Soc. 71 (1965), 742–746.


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