A RELATION BETWEEN A THEOREM OF BOHR AND SIDON SETS

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1. Introduction. In 1913, Bohr [1] proved the following theorem for Dirichlet series: if

\[ f(\sigma + it) = \sum_{n=1}^{\infty} c(n)n^{-\sigma-it} \]

and if \(|f(\sigma+it)| \leq 1\) for all \(\sigma > 0\), then

\[ \sum_{p} |c(p)| \leq 1, \]

the sum in (2) extending over all primes.

A set of positive integers \(E\) will be called a Bohr set if there is a finite constant \(B\) such that for every function \(f\) as in (1)

\[ \sum_{n \in E} |c(n)| \leq B. \]

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It is easily seen that $E$ is a Bohr set if and only if for every finite sum $f(t) = \sum c(n)n^{-it}$

$$
\sum_{n \in E} |c(n)| \leq B \sup_{-\infty < t < \infty} |f(t)|.
$$

Let $G$ be a compact Abelian group and $E$ a subset of its dual group $\Gamma$. An $E$-*polynomial* is a trigonometric polynomial, $F$, such that $\hat{F}(\gamma) = 0$ for $\gamma \in E$ where

$$
\hat{F}(\gamma) = \int_G F(x)\gamma(-x) \, dx, \quad \gamma \in \Gamma.
$$

Here $dx$ is normalized Haar measure on $G$. $E$ is called a *Sidon set* if there is a finite constant $B$ such that

$$
\sum_{\gamma \in E} |\hat{F}(\gamma)| \leq B \sup_{x \in G} |F(x)| = B\|F\|_\infty
$$

for every $E$-polynomial $F$.

Let $T^\omega$ be the direct product of a countably infinite collection of circles. $T^\omega$ is a compact Abelian group with dual group $Z^\omega$. Each $\gamma \in Z^\omega$ is given by a sequence of integers $\{\alpha_n\}$ where only a finite number of the $\alpha_n$ are not zero. $M(T^\omega)$ is the space of regular Borel measures, $\mu$, on $T^\omega$ with finite total variation $\|\mu\|$. $\hat{\mu}$ is the Fourier-Stieltjes transform of $\mu$.

In this note we give a characterization of Bohr sets in terms of Sidon sets in $Z^\omega$ and certain measures on $T^\omega$. It is then possible to obtain a sufficient arithmetic condition for Bohr sets.

2. The relation between Bohr sets and Sidon sets. $P$ will denote the positive cone of $Z^\omega$. Let $p_1, p_2, \cdots$ be the primes. If $n$ is an integer and $n = \prod p_i^{\alpha_i}$, then we associate $n$ with the element $\gamma_n = (\alpha_1, \alpha_2, \cdots)$ of $P$. For a set of positive integers $E$, $\hat{E} = \{ \gamma_n : n \in E \}$.

To a function $f(t) = \sum c(n)n^{-it}$ we associate the function $F(x) = \sum c(n)\gamma_n(x)$ on $T^\omega$. Bohr noticed the following: if $\phi: (-\infty, \infty) \to T^\omega$ by

$$
\phi(t) = (\exp(-it \log p_1), \exp(-it \log p_2), \cdots)
$$

then $\gamma_n(\phi(t)) = n^{-it}$ so that $F(\phi(t)) = f(t)$. Now since $\{ \log p_j \}$ is linearly independent over the integers, $\phi(-\infty, \infty)$ is dense in $T^\omega$. Thus

$$
\|F\|_\infty = \sup_{-\infty < t < \infty} |f(t)|.
$$
THEOREM. A set of positive integers $E$ is a Bohr set if and only if
(a) $\hat{E}$ is a Sidon set in $T^*$, and
(b) there is a measure $\mu \in M(T^*)$ such that

$$
\hat{\mu}(\gamma) = \begin{cases} 
1 & \text{if } \gamma \in \hat{E}, \\
0 & \text{if } \gamma \in P - \hat{E}.
\end{cases}
$$

PROOF. Let $F(x) = \sum \hat{\gamma} \gamma_n(x)$ be a $P$-polynomial and let $f(t) = \sum \hat{\gamma} \gamma_n n^{-it}$.

If $E$ is a Bohr set then by (7)

$$
\sum_{\gamma \in \hat{E}} |\hat{\gamma}| = \sum_{n \in \mathbb{Z}} |\hat{\gamma}_n| \leq B \sup_{-\infty < t < \infty} |f(t)| = B\|F\|_\infty.
$$

Thus if $b$ is a function on $\hat{E}$ and $|b(\gamma)| \leq 1$ then $L(f) = \sum_{\gamma \in \hat{E}} b(\gamma)\hat{\gamma}(\gamma)$ is a bounded linear functional on the $P$-polynomials with norm at most $B$. By the Hahn-Banach and Riesz representation theorems there is a measure $\mu \in M(T^*)$ with

$$
\hat{\mu}(\gamma) = b(\gamma), \quad \gamma \in \hat{E},
$$

$$
= 0, \quad \gamma \in P - \hat{E}.
$$

By [3, Theorem 5.7.3], $\hat{E}$ is a Sidon set; by taking $b \equiv 1$ we obtain the measure for (b).

Conversely suppose (a) and (b) are true for $E$ and let $f(t) = \sum c(n) n^{-it}$ be a finite sum. By (a) and the proof of [3, Theorem 5.7.3] there is $\nu \in M(T^*)$ with $\|\nu\| \leq B$ ($B$ depends only on $E$) and $c(n)\nu(\gamma_n) = |c(n)|$ for $n \in E$. Let $\mu$ be as in (b) and $*$ denote ordinary convolution. Then

$$
\sum_{n \in E} |c(n)| \leq B\|\mu\| \|F\|_\infty \
\leq B' \sup_{-\infty < t < \infty} |f(t)|.
$$

COROLLARY. Let $E = \{n_1, n_2, \cdots\}$ be a set of positive integers satisfying
(c) $\{\log n_j\}$ are linearly independent over the integers, and
(d) if $n$ is a positive integer, $\{\beta_j\}$ is a collection of integers, $\sum \beta_j = 1$, and $n = \prod n_j^k$ then $n \in E$.

Then $E$ is a Bohr set.

PROOF. It follows from (c) that if $k_1 < k_2 < \cdots < k_s$ then $0 \neq \pm \gamma_{n_{k_1}} \pm \gamma_{n_{k_2}} \pm \cdots \pm \gamma_{n_{k_s}}$. Thus by [2, Theorem 1.5], $\hat{E}$ is a Sidon set.
Let $H = \{ \gamma \in \mathbb{Z}^n : \gamma = \sum \beta_i \gamma_{n_i}, \beta_i \text{ integers}, \sum \beta_i = 1 \}$. $H$ is a coset of a subgroup of $\mathbb{Z}^n$ and by (d) $\tilde{E} = H' \cap P$. By [3, p. 60] there is $\mu \in M(T^n)$ such that $\tilde{\mu}$ is the characteristic function of $H'$. $\mu$ satisfies condition (b) of the theorem.

3. **Examples.** The corollary shows that there are Bohr sets which are not the finite union of sets with pairwise relatively prime elements. For example, $p_1 p_2, p_1 p_3, p_1 p_4, p_2 p_3, p_4 p_5, p_5 p_6, \ldots$. It is known [3, p. 126] that every infinite subset of a discrete group contains an infinite Sidon subset. However this is not true of Bohr sets.

**Example.** Let $F = \{ n_j = (p_1 p_2 \cdots p_j) ! \}$. Then $F$ contains no infinite Bohr subset.

In fact $F$ contains no infinite subset for which there is a measure satisfying (8). For suppose $E = \{ n_j, n_{j+1}, \ldots \}$ and $\hat{\mu}$ satisfies (8). Let $\mu_\gamma$ be the translation of $\mu$ such that

$$\hat{\mu}_\gamma (\gamma) = \hat{\mu} (\gamma + \gamma_{n_j}).$$

$\{ \mu_\gamma \}$ has a weak star convergent subsequence to a measure $\nu \in M(T^n)$ which by a lemma of Helson [3, p. 66] must be singular with respect to Haar measure.

But this is impossible since it is easily seen that

$$\hat{\nu} (\gamma) = 1 \quad \text{if} \quad \gamma = 0,$$

$$= 0 \quad \text{if} \quad \gamma \neq 0$$

so that $\nu$ must be the Haar measure.

This example also shows that the corollary is false without (d).

**BIBLIOGRAPHY**

